

COMPUTING HOMOMORPHISMS BETWEEN HOLONOMIC D -MODULES

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ABSTRACT. Let $K \subseteq \mathbb{C}$ be a subfield of the complex numbers, and let D be the ring of K -linear differential operators on $R = K[x, \dots, x_n]$. If M and N are holonomic left D -modules we present an algorithm that computes explicit generators for the finite dimensional vector space $\text{Hom}_D(M, N)$. This enables us to answer algorithmically whether two given holonomic modules are isomorphic. More generally, our algorithm can be used to get explicit generators for $\text{Ext}_D^i(M, N)$ for any i .

1. INTRODUCTION

Let $D = D_n = K\langle x_1, \dots, x_n, \partial_1, \dots, \partial_n \rangle$ denote the n -th Weyl algebra over a computable subfield $K \subset \mathbb{C}$, i.e. elements of K can be represented with a finite set of data, their sums, products and quotients can be calculated in a finite number of steps, and there is a finite procedure that determines whether a given expression of elements of K is zero or not. Let $\text{Hom}_D(M, N)$ denote the set of left D -module maps between two left D -modules M and N . Then $\text{Hom}_D(M, N)$ is a K -vector space and can also be regarded as the solutions of M inside N in the following way: Given a presentation $M \simeq D^{r_0}/D \cdot \{L_1, \dots, L_{r_1}\}$, let S denote the system of vector-valued linear partial differential equations,

$$S = \{L_1 \bullet f = \dots = L_{r_1} \bullet f = 0\},$$

and let $\text{Sol}(S; N)$ denote the N -valued solutions $f \in N^{r_0}$ to S . Then the homomorphism space $\text{Hom}_D(D^{r_0}/D \cdot \{L_1, \dots, L_{r_1}\}, N)$ is isomorphic to the solution space $\text{Sol}(S; N)$ where the identification is as follows. A homomorphism φ in $\text{Hom}_D(D^{r_0}/D \cdot \{L_1, \dots, L_{r_1}\}, N)$ corresponds to the solution $[\varphi(e_1), \dots, \varphi(e_{r_0})]^T \in N^{r_0}$ of S , while a solution $f = [f_1, \dots, f_{r_0}]^T \in N^{r_0}$ of S corresponds to the homomorphism which sends e_i to f_i .

If M and N are holonomic, then the set $\text{Hom}_D(M, N)$ as well as the higher derived functors $\text{Ext}_D^i(M, N)$ are finite-dimensional K -vector spaces. In this paper, we give algorithms that compute explicit bases for $\text{Hom}_D(M, N)$ and $\text{Ext}_D^i(M, N)$ in this situation. Our algorithms are a refinement of algorithms given in [12], which were designed to compute the dimensions of $\text{Hom}_D(M, N)$ and $\text{Ext}_D^i(M, N)$ over K . Algebraically, our problem of computing a basis of homomorphisms is easy to describe. Namely, since a map of left D -modules from M to N is uniquely determined by the images of a set of generators of M , we must simply determine which sets of elements of N constitute legal choices for the images of a homomorphism (of

a fixed set of generators of M). It is perhaps surprising that this is a difficult computation. One of the reasons is that $\text{Hom}_D(M, N)$ lacks any D -module structure in general and is just a K -vector space.

In recent years, one of the fundamental advances in computational D -modules has been the development of algorithms by Oaku and Takayama [9, 10] to compute the derived restriction modules $\text{Tor}_i^D(D/\{x_1, \dots, x_d\} \cdot D, M)$ and derived integration modules $\text{Tor}_i^D(D/\{\partial_1, \dots, \partial_d\} \cdot D, M)$ of a holonomic D -module M to a linear subspace $x_1 = \dots = x_d = 0$. We give a summary of these algorithms in the appendix. These algorithms have been the basis for local cohomology and de Rham cohomology algorithms [11, 17] and have been extended to algorithms for derived restriction and integration of complexes with holonomic cohomology by the second author [16].

Similarly, the algorithm of [12] to compute the dimensions of $\text{Hom}_D(M, N)$ and $\text{Ext}_D^i(M, N)$ is also based on restriction by using isomorphisms of Kashiwara and Björk [1, 7]. These isomorphisms are,

$$(1) \quad \text{Ext}_D^i(M, N) \cong \text{Tor}_{n-i}^D(\text{Ext}_D^n(M, D), N),$$

which turns an Ext computation into a Tor computation and

$$(2) \quad \text{Tor}_j^D(M', N) \simeq \text{Tor}_j^{D^{2n}}(D_{2n}/\{x_i - y_i, \partial_i + \delta_i\}_{i=1}^n \cdot D_{2n}, \tau(M') \boxtimes N),$$

which turns a Tor computation into a twisted restriction computation in twice as many variables (an explanation of the notation used above can be found in Section 5).

In this paper, we will obtain an algorithm for computing an explicit basis of $\text{Ext}_D^i(M, N)$ by analyzing the isomorphisms (1) and (2) and making them compatible with the restriction algorithm. In Section 2, we present a proof of isomorphism (1) adapted from [1]. In Section 3, we give an algorithm for computing $\text{Hom}_D(M, N)$ in the case $N = K[x_1, \dots, x_n]$, which is used to compute polynomial solutions of a system S . In Section 4, we give an algorithm for the case $N = K[x_1, \dots, x_n][f^{-1}]$, which can be used to compute rational solutions of S . In Section 5, we give our main result, which is an algorithm to compute $\text{Hom}_D(M, N)$ for general holonomic modules M, N . In Section 6, we give a companion algorithm which computes $\text{Ext}_D^i(M, N)$ and their representation in terms of Yoneda Ext groups. In Section 7, we give an algorithm to determine whether M and N are isomorphic and if so to find an isomorphism. We also give an algorithm to compute the endomorphism ring $\text{End}_D(M)$, the algebraic group $\text{Iso}_D(M)$, and some of their basic properties. In the appendix, we review the restriction and integration algorithms. The reader may refer to Algorithm 8.6 for a discussion of the restriction algorithm and the V -filtration, and to Algorithm 8.8 for a discussion of the integration algorithm and the \tilde{V} -filtration. Finally, the algorithms described in this paper have been implemented in Macaulay 2 [6].

1.1. Notation. Throughout we shall denote the ring of polynomials $K[x_1, \dots, x_n]$ by $K[\mathbf{x}]$, the ring of polynomials $K[\partial_1, \dots, \partial_n]$ by $K[\partial]$, and the ring $K[\mathbf{x}]\langle\partial\rangle$ of K -linear differential operators on $K[\mathbf{x}]$ by D .

Let us also explain the notation we will use to write maps of left or right D -modules. As usual, maps between finitely generated modules will be represented by matrices, but some attention has to be given to the order in which elements are

multiplied due to the noncommutativity of D . Let us denote the identity matrix of size r by id_r , and similarly the identity map on the module M by id_M .

Given an $r \times s$ matrix $A = [a_{ij}]$ with entries in D , we get a map of free left D -modules,

$$D^r \xrightarrow{\cdot A} D^s \quad : \quad [\ell_1, \dots, \ell_r] \mapsto [\ell_1, \dots, \ell_r] \cdot A,$$

where D^r and D^s are regarded as modules of row vectors, and the map is matrix multiplication. Under this convention, the composition of maps $D^r \xrightarrow{\cdot A} D^s$ and $D^s \xrightarrow{\cdot B} D^t$ is the map $D^r \xrightarrow{\cdot AB} D^t$ where AB is usual matrix multiplication.

In general, suppose M and N are left D -modules with presentations D^r/M_0 and D^s/N_0 . Then the matrix A induces a left D -module map between M and N , denoted $(D^r/M_0) \xrightarrow{\cdot A} (D^s/N_0)$, precisely when $L \cdot A \in N_0$ for all row vectors $L \in M_0$. This condition need only be checked for a generating set of M_0 . Conversely, any map of left D -modules between M and N can be represented by some matrix A in the manner above.

Now let us discuss maps of right D -modules. The $r \times s$ matrix A also defines a map of right D -modules in the opposite direction,

$$(D^s)^T \xrightarrow{A \cdot} (D^r)^T \quad : \quad [\ell'_1, \dots, \ell'_s]^T \mapsto A \cdot [\ell'_1, \dots, \ell'_s]^T,$$

where the superscript- T means to regard the free modules $(D^s)^T$ and $(D^r)^T$ as consisting of column vectors. This map is equivalent to the map obtained by applying $\text{Hom}_D(-, D)$ to $D^r \xrightarrow{\cdot A} D^s$, thus $(D^s)^T$ may alternatively be regarded as the dual module $\text{Hom}_D(D^s, D)$. We will suppress the superscript- T when the context is clear. As before, the matrix A induces a right D -module map between right D -modules $N' = (D^s)^T/N'_0$ and $M' = (D^r)^T/M'_0$ whenever $A \cdot L \in M'_0$ for all column vectors $L \in N'_0$. We denote the map by $(D^s)^T/N'_0 \xrightarrow{A \cdot} (D^r)^T/M'_0$.

1.2. Left-right correspondence. The category of left D -modules is equivalent to the category of right D -modules, and for convenience, we will sometimes prefer to work in one category rather than the other – for instance, we will phrase all algorithms in terms of left D -modules. In the Weyl algebra, the correspondence is given by the algebra involution

$$D \xrightarrow{\tau} D \quad : \quad x^\alpha \partial^\beta \mapsto (-\partial)^\beta x^\alpha.$$

The map τ is called the standard transposition or adjoint operator. Given a left D -module D^r/M_0 , the corresponding right D -module is

$$\tau\left(\frac{D^r}{M_0}\right) := \frac{D^r}{\tau(M_0)}, \quad \tau(M_0) = \{\tau(L) | L \in M_0\},$$

Similarly, given a homomorphism of left D -modules $\phi : D^r/M_0 \rightarrow D^s/N_0$ defined by right multiplication by the $r \times s$ matrix $A = [a_{ij}]$, the corresponding homomorphism of right D -modules $\tau(\phi) : D^r/\tau(M_0) \rightarrow D^s/\tau(N_0)$ is defined by right multiplication by the $s \times r$ matrix $\tau(A) := [\tau(a_{ij})]^T$. The map τ is used similarly to go from right to left D -modules. For more details, see [12].

2. BASIC ISOMORPHISM

The following identification, which we take from Björk [1], is our main theoretical tool to explicitly compute homomorphisms of holonomic D -modules.

Theorem 2.1. [1] *Let M and N be holonomic left D -modules. Then*

$$(3) \quad \text{Ext}_D^i(M, N) \cong \text{Tor}_{n-i}^D(\text{Ext}_D^n(M, D), N).$$

Proof. Since it will be useful to us later, we give the main steps of the proof here. The interesting bit of the construction is the transformation of a Hom into a tensor product. The presentation is adapted from [1]. Let X^\bullet be a free resolution of M ,

$$X^\bullet : 0 \rightarrow D^{r-a} \xrightarrow{\cdot M_{-a+1}} \dots \rightarrow D^{r-1} \xrightarrow{\cdot M_0} D^{r_0} \rightarrow M \rightarrow 0$$

We may assume it is of finite length by virtue of Hilbert's syzygy theorem – namely, Schreyer's proof and method carries over to D (see e.g. [2]). The dual of X^\bullet is the complex of right D -modules,

$$\text{Hom}_D(X^\bullet, D) : 0 \leftarrow \underbrace{(D^{r-a})^T}_{\text{degree } a} \xleftarrow{M_{-a+1} \cdot} \dots \leftarrow (D^{r-1})^T \xleftarrow{M_0 \cdot} \underbrace{(D^{r_0})^T}_{\text{degree } 0} \leftarrow 0$$

Since $\text{Hom}_D(D^r, D) \otimes_D N \simeq \text{Hom}_D(D^r, N)$, we see that $\text{Hom}_D(X^\bullet, D) \otimes_D N \simeq \text{Hom}_D(X^\bullet, N)$, whose cohomology groups are by definition $\text{Ext}_D^i(M, N)$. Now as is customary, replace N by a free resolution Y^\bullet , which we may also take to be of finite length,

$$(4) \quad Y^\bullet : 0 \rightarrow D^{s-b} \xrightarrow{\cdot N_{-b+1}} \dots \rightarrow D^{s-1} \xrightarrow{\cdot N_0} D^{s_0} \rightarrow N \rightarrow 0$$

We get the double complex $\text{Hom}_D(X^\bullet, D) \otimes_D Y^\bullet$,

$$(5) \quad \begin{array}{ccccccc} & 0 & & 0 & & 0 & \\ & \uparrow & & \uparrow & & \uparrow & \\ 0 \leftarrow & (D^{r-a})^T \otimes_D D^{s_0} & \xleftarrow{(M_{-a+1} \cdot) \otimes \text{id}_{s_0}} & \dots & \xleftarrow{(D^{r-1})^T \otimes_D D^{s_0}} & \xleftarrow{(M_0 \cdot) \otimes \text{id}_{s_0}} & (D^{r_0})^T \otimes_D D^{s_0} \leftarrow 0 \\ & \uparrow & & & \uparrow & & \uparrow \\ & (-\text{id}_{r-a})^a \otimes (\cdot N_0) & & & -\text{id}_{r-1} \otimes (\cdot N_0) & & \text{id}_{r_0} \otimes (\cdot N_0) \\ 0 \leftarrow & (D^{r-a})^T \otimes_D D^{s-1} & \xleftarrow{(M_{-a+1} \cdot) \otimes \text{id}_{s-1}} & \dots & \xleftarrow{(D^{r-1})^T \otimes_D D^{s-1}} & \xleftarrow{(M_0 \cdot) \otimes \text{id}_{s-1}} & (D^{r_0})^T \otimes_D D^{s-1} \leftarrow 0 \\ & \uparrow & & & \uparrow & & \uparrow \\ & \vdots & & & \vdots & & \vdots \\ & \uparrow & & & \uparrow & & \uparrow \\ & (-\text{id}_{r-a})^a \otimes (\cdot N_{-b+1}) & & & -\text{id}_{r-1} \otimes (\cdot N_{-b+1}) & & \text{id}_{r_0} \otimes (\cdot N_{-b+1}) \\ 0 \leftarrow & (D^{r-a})^T \otimes_D D^{s-b} & \xleftarrow{(M_{-a+1} \cdot) \otimes \text{id}_{s-b}} & \dots & \xleftarrow{(D^{r-1})^T \otimes_D D^{s-b}} & \xleftarrow{(M_0 \cdot) \otimes \text{id}_{s-b}} & (D^{r_0})^T \otimes_D D^{s-b} \leftarrow 0 \\ & \uparrow & & & \uparrow & & \uparrow \\ & 0 & & & 0 & & 0 \end{array}$$

Since the columns of the double complex are exact except for at positions in the top row, it follows that the cohomology of the total complex equals the cohomology

of the complex induced on the table of E_1 terms (vertical cohomologies),

$$(6) \quad 0 \leftarrow \underbrace{\mathrm{Hom}_D(D^{r-a}, N)}_{\text{degree } a} \xleftarrow{\mathrm{Hom}_D((M_{-a+1}\cdot), N)} \cdots \xleftarrow{\mathrm{Hom}_D((M_0\cdot), N)} \underbrace{\mathrm{Hom}_D(D^{r_0}, N)}_{\text{degree } 0} \leftarrow 0$$

As stated earlier, these cohomology groups are $\mathrm{Ext}_D^i(M, N)$.

On the other hand, since M is holonomic, the complex $\mathrm{Hom}_D(X^\bullet, D)$ is exact except in degree n , where its cohomology is by definition $\mathrm{Ext}_D^n(M, D)$. Hence the rows of the double complex are also exact except at positions in the n -th column, i.e. the column containing terms $(D^{r-n} \otimes_D (-))$. It follows that the cohomology of the total complex also equals the cohomology of the complex induced on the other table of E_1 terms (horizontal cohomologies), which in this case is

$$(7) \quad 0 \rightarrow \mathrm{Ext}_D^n(M, D) \otimes_D D^{s-b} \rightarrow \cdots \xrightarrow{\mathrm{id}_{\mathrm{Ext}_D^n(M, D)} \otimes (\cdot N_0)} \mathrm{Ext}_D^n(M, D) \otimes_D D^{s_0} \rightarrow 0$$

By definition, the above complex has cohomology groups $\mathrm{Tor}_j^D(\mathrm{Ext}_D^n(M, D), N)$, which establishes the identification. \square

In the next few sections, our goal will be to compute an explicit basis of cohomology classes of the complex (6). In particular, the cohomology in degree 0 corresponds explicitly to $\mathrm{Hom}_D(M, N)$ because any map $\psi \in \mathrm{Hom}_D(D^{r_0}, N)$ which is in the degree 0 kernel, i.e. in

$$(8) \quad H^0(\mathrm{Hom}_D(D^{r-1}, N) \xleftarrow{\mathrm{Hom}_D((M_0\cdot), N)} \underbrace{\mathrm{Hom}_D(D^{r_0}, N)}_{\text{degree } 0} \leftarrow 0),$$

factors through $M \simeq D^{r_0}/M_0$, hence defines a homomorphism $\bar{\psi} : M \rightarrow N$. The reason why it is hard to compute these cohomology classes is that the modules $\mathrm{Hom}_D(D^{r_i}, N)$ in the complex (6) are left D -modules while the maps $\mathrm{Hom}_D((M_i\cdot), N)$ are not maps of left D -modules. In the next few sections, we will explain how the ingredients of the proof of Theorem 2.1 can be combined with the restriction algorithm to compute the desired cohomology classes.

3. POLYNOMIAL SOLUTIONS

In this section, we give an algorithm to compute $\mathrm{Hom}_D(M, K[\mathbf{x}])$ for holonomic M . This vector space is more efficiently computed by Gröbner deformations as described in [12], but we wish to discuss this special case in order to introduce the general methodology.

For $N = K[\mathbf{x}]$, the isomorphism (3) of Theorem 2.1 specializes to

$$(9) \quad \mathrm{Ext}_D^i(M, K[\mathbf{x}]) \simeq \mathrm{Tor}_{n-i}^D(\mathrm{Ext}_D^n(M, D), K[\mathbf{x}]).$$

In this case, the proof of Theorem 2.1 also leads directly to an algorithm. As a D -module, the polynomial ring has the presentation $K[\mathbf{x}] \simeq D/D \cdot \{\partial_1, \dots, \partial_n\}$ and can be resolved by the Koszul complex,

$$\mathcal{K}^\bullet : 0 \rightarrow \underbrace{D}_{\text{degree } n} \xrightarrow{\cdot[(-1)^{n-1}\partial_n, \dots, \partial_1]} D^n \rightarrow \cdots \rightarrow D^n \xrightarrow{\cdot \begin{bmatrix} \partial_1 \\ \vdots \\ \partial_n \end{bmatrix}} \underbrace{D}_{\text{degree } 0} \rightarrow 0.$$

The complex (7) whose cohomology computes $\mathrm{Tor}_{n-i}^D(\mathrm{Ext}_D^n(M, D), K[\mathbf{x}])$ then specializes to $\mathrm{Ext}_D^n(M, D) \otimes_D \mathcal{K}^\bullet$ and is equivalently the derived integration complex of $\mathrm{Ext}_D^n(M, D)$ in the category of right D -modules. Oaku and Takayama's integration algorithm can now be applied to obtain a basis of explicit cohomology classes in $H^n(\mathrm{Ext}_D^n(M, D) \otimes_D \mathcal{K}^\bullet) \simeq \mathrm{Tor}_n^D(\mathrm{Ext}_D^n(M, D), K[\mathbf{x}])$. These classes can then be transferred via the double complex (5) to cohomology classes in the complex (8), where they represent homomorphisms in $\mathrm{Hom}_D(M, K[\mathbf{x}])$. The method and details are probably best illustrated through an example.

Example 3.1. Consider the Gelfand-Kapranov-Zelevinsky hypergeometric system $M_A(\beta)$ associated to the matrix $A = \{1, 2\}$ and parameter vector $\beta = \{5\}$, i.e. the D -module associated to the equations,

$$u = \theta_1 + 2\theta_2 - 5 \quad v = \partial_1^2 - \partial_2$$

Here, θ_i stands for the operator $x_i \partial_i$.

A resolution for $M_A(\beta)$ is

$$X^\bullet : 0 \rightarrow D^1 \xrightarrow{\cdot[-v \ u+2]} D^2 \xrightarrow{\cdot\begin{bmatrix} u \\ v \end{bmatrix}} D^1 \rightarrow 0$$

while a resolution for $K[x_1, x_2]$ is the Koszul complex,

$$\mathcal{K}^\bullet : 0 \rightarrow D \xrightarrow{\cdot[\partial_1, \partial_2]} D^2 \xrightarrow{\cdot\begin{bmatrix} \partial_2 \\ -\partial_1 \end{bmatrix}} D \rightarrow 0$$

The augmented double complex $\mathrm{Hom}_D(X^\bullet, D) \otimes_D \mathcal{K}^\bullet$ is

$$\begin{array}{ccccccc}
& & K[x_1, x_2] & \xleftarrow{\cdot[-v \ u+2] \bullet} & K[x_1, x_2]^2 & \xleftarrow{\cdot\begin{bmatrix} u \\ v \end{bmatrix} \bullet} & K[x_1, x_2] \\
& & \uparrow & & \uparrow & & \uparrow \\
& & D^1 & \xleftarrow{\cdot[-v \ u+2]} & D^2 & \xleftarrow{\cdot\begin{bmatrix} u \\ v \end{bmatrix}} & D^1 \\
& & \uparrow & & \uparrow & & \uparrow \\
\mathrm{Ext}_D^n(M_A(\beta), D) & \xleftarrow{\cdot\begin{bmatrix} \partial_2 \\ -\partial_1 \end{bmatrix}} & D^1 & \xleftarrow{\cdot\begin{bmatrix} \partial_2 & 0 \\ 0 & \partial_2 \\ -\partial_1 & 0 \\ 0 & -\partial_1 \end{bmatrix}} & D^2 & \xleftarrow{\cdot\begin{bmatrix} u & 0 \\ v & 0 \\ 0 & u \\ 0 & v \end{bmatrix}} & D^2 \\
& & \uparrow & & \uparrow & & \uparrow \\
\mathrm{Ext}_D^n(M_A(\beta), D)^2 & \xleftarrow{\cdot[\partial_1 \ \partial_2]} & D^2 & \xleftarrow{\cdot\begin{bmatrix} -v & u+2 & 0 & 0 \\ 0 & 0 & -v & u+2 \end{bmatrix}} & D^4 & \xleftarrow{\cdot\begin{bmatrix} \partial_1 & 0 & \partial_2 & 0 \\ 0 & \partial_1 & 0 & \partial_2 \end{bmatrix}} & D^2 \\
& & \uparrow & & \uparrow & & \uparrow \\
\mathrm{Ext}_D^n(M_A(\beta), D) & \xleftarrow{\cdot\begin{bmatrix} \partial_1 & \partial_2 \end{bmatrix}} & \boxed{D^1} & \xleftarrow{\cdot[-v \ u+2]} & D^2 & \xleftarrow{\cdot\begin{bmatrix} u \\ v \end{bmatrix}} & D^1
\end{array}$$

Here, we interpret an element of a module in the above diagram as a column vector for purposes of the horizontal maps and as a row vector for purposes of the vertical maps. The induced complex at the left-hand wall is the derived integration to the origin of $\mathrm{Ext}_D(M_A(\beta), D)$ in the category of right D -modules. Applying the integration algorithm, we find that the cohomology at the module $\boxed{D^1}$ in the bottom left-hand corner is 1-dimensional and spanned by the residue class of

$$L_{1,0} = -(2x_1^5 x_2 - 40x_1^3 x_2^2 + 120x_1 x_2^3) \partial_1 - (x_1^6 - 30x_1^4 x_2 + 180x_1^2 x_2^2 - 120x_2^3).$$

We lift this class to a cohomology class of the complex induced at the top row via a “transfer” sequence in the total complex given schematically by

$$\begin{array}{ccc}
 & D^2 \xleftarrow{[u \atop v]} D^1 \ni L_{1,2} & \\
 & \uparrow \left[\begin{array}{cc} \partial_2 & 0 \\ 0 & \partial_2 \\ -\partial_1 & 0 \\ 0 & -\partial_1 \end{array} \right] & \\
 D^2 \xleftarrow{[-v \quad u+2 \quad 0 \quad 0 \atop 0 \quad 0 \quad -v \quad u+2]} D^4 \ni L_{1,1} & & \\
 \uparrow \cdot [\partial_1 \quad \partial_2] & & \\
 D^1 \ni L_{1,0} & &
 \end{array}$$

In other words, $L_{1,1}$ is obtained by taking the image of $L_{1,0}$ under the vertical map and then a pre-image under the horizontal map, and similarly for $L_{1,2}$. We find that,

$$\begin{aligned}
 L_{1,1} &= \begin{bmatrix} 2x_1^5x_2 - 40x_1^3x_2^2 + 120x_1x_2^3 \\ -(x_1^5 - 20x_1^3x_2 + 60x_1x_2^2) \\ -(x_1^6 - 20x_1^4x_2 + 60x_1^2x_2^2) \\ (x_1^5 - 20x_1^3x_2 + 60x_1x_2^2)\partial_1 + (10x_1^4 - 120x_1^2x_2 + 120x_2^2) \end{bmatrix}, \\
 L_{1,2} &= [x_1^5 - 20x_1^3x_2 + 60x_1x_2^2].
 \end{aligned}$$

The space of polynomial solutions is spanned by the residue class of $L_{1,2}$ in $K[x_1, x_2]$, which is $x_1^5 - 20x_1^3x_2 + 60x_1x_2^2$.

Remark 3.2. The transfer sequence above is used to show that Tor is a balanced functor in Weibel [18]. A generalization of the transfer sequence is also used by the second author to compute the cup product structure for de Rham cohomology of the complement of an affine variety in [17].

From a practical standpoint, the method outlined above is not quite the final story. The detail we have left out is how Oaku and Takayama’s integration algorithm actually computes the cohomology classes of a Koszul complex such as $\text{Ext}_D^n(M, D) \otimes_D \mathcal{K}^\bullet$. Their algorithm does not compute these classes directly. Rather, their method (phrased in terms of right D -modules) is to first compute a \tilde{V} -strict resolution Z^\bullet of $\text{Ext}_D^n(M, D)$. Then they give a technique to compute explicitly the cohomology of $Z^\bullet \otimes_D K[\mathbf{x}]$. This complex is quasi-isomorphic to $\text{Ext}_D^n(M, D) \otimes_D \mathcal{K}^\bullet$, and cohomology classes are transferred by setting up another double complex $Z^\bullet \otimes_D \mathcal{K}^\bullet$. Thus, our method as described to compute polynomial solutions requires two transfers via two double complexes.

Given the true nature of the integration algorithm, the two transfers can be collapsed into a single step. Namely, we start with $\text{Hom}_D(X^\bullet, D)$,

$$\text{Hom}_D(X^\bullet, D) : 0 \leftarrow \dots \xleftarrow{M_{-n}} \underbrace{(D^{r-n})^T}_{\text{degree } n} \xleftarrow{M_{-n+1}} \dots \xleftarrow{M_0} \underbrace{(D^{r_0})^T}_{\text{degree } 0} \leftarrow 0$$

which is exact except in cohomological degree n because M is holonomic. We are interested in explicit cohomology classes for $H^0(\text{Hom}_D(X^\bullet, D) \otimes_D \mathcal{K}[\mathbf{x}])$. To obtain

them, we replace $\text{Hom}_D(X^\bullet, D)$ with a quasi-isomorphic \tilde{V} -adapted resolution E^\bullet along with an explicit quasi-isomorphism π_\bullet from E^\bullet to $\text{Hom}_D(X^\bullet, D)$. That is, we make a map π_n from a free module $(D^{s-n})^T$ onto some choice of generators of $\ker(M_{-n}\cdot)$, take the pre-image P of $\text{im}(M_{-n+1}\cdot)$ under π_n , and compute a \tilde{V} -adapted resolution E^\bullet of D^{s-n}/P . Schematically,

$$\begin{array}{ccccccc}
0 \leftarrow & \underbrace{(D^{s-n})^T}_P & \xleftarrow{\quad} & (D^{s-n})^T & \xleftarrow{N_{-n+1}\cdot} & (D^{s-n+1})^T \dots & \xleftarrow{N_0\cdot} (D^{s_0})^T \leftarrow (D^{s_1})^T \leftarrow \dots \\
& & & \downarrow \pi_n & & \vdots & \downarrow \vdots \\
0 \leftarrow \dots \leftarrow & (D^{r-n-1})^T & \xleftarrow{M_{-n}\cdot} & \underbrace{(D^{r-n})^T}_{\text{degree } n} & \xleftarrow{M_{-n+1}\cdot} & (D^{r-n+1})^T \dots & \xleftarrow{M_0\cdot} \underbrace{(D^{r_0})^T}_{\text{degree } 0} \leftarrow 0
\end{array}$$

Using the integration algorithm, the cohomology classes of the top row can now be computed. In order to transfer them to $\text{Hom}_D(X^\bullet, D) \otimes_D K[\mathbf{x}]$, a chain map lifting π_n is computed and utilized as suggested by the dashed arrows. We summarize the algorithm as follows. To keep computations in terms of left D -modules, we make use of the transposition τ at various places. Applying τ to the polynomial ring gives the top differential forms $\Omega = D/\{\partial_1, \dots, \partial_n\} = \tau(K[\mathbf{x}])$.

Algorithm 3.3. [Polynomial solutions by duality]

INPUT: $\{L_1, \dots, L_{r_1}\} \subset D^{r_0}$ such that $M = (D^{r_0}/D \cdot \{L_1, \dots, L_{r_1}\})$ is holonomic.

OUTPUT: The polynomial solutions $R \in K[\mathbf{x}]^{r_0}$ of the system of differential equations given by $L_i \bullet R = 0$, $i = 1, \dots, r_1$.

1. Compute a free resolution X^\bullet of M of length $n + 1$. Let its part in cohomological degree $-n$ be denoted:

$$\dots \rightarrow D^{r-n-1} \xrightarrow{\cdot M_{-n}} D^{r-n} \xrightarrow{\cdot M_{-n+1}} D^{r-n+1} \rightarrow \dots$$

2. Form the complex $\tau(\text{Hom}_D(X^\bullet, D))$ obtained by dualizing X^\bullet and then applying the standard transposition. Its part in cohomological degree n now looks like:

$$\dots \leftarrow D^{r-n-1} \xleftarrow{\cdot \tau(M_{-n})} D^{r-n} \xleftarrow{\cdot \tau(M_{-n+1})} D^{r-n+1} \leftarrow \dots$$

3. Compute a surjection $\pi_n : D^{s-n} \twoheadrightarrow \ker(\cdot \tau(M_{-n}))$, and find the pre-image $\tau(P) := \pi_n^{-1}(\text{im}(\cdot \tau(M_{-n+1})))$. This yields the presentation $D^{s-n}/\tau(P) \simeq \tau(\text{Ext}_D^n(M, D))$.

4. Compute the derived integration module $H^0((\Omega \otimes_D^L (D^{s-n}/\tau(P)))[n])$ using Algorithm 8.8. In particular, this algorithm produces

- (i.) A \tilde{V} -strict free resolution E^\bullet of $D^{s-n}/\tau(P)$ of length $n + 1$,

$$E^\bullet : 0 \leftarrow \underbrace{D^{s-n}}_{\text{degree } n} \leftarrow D^{s-n+1} \leftarrow \dots \leftarrow D^{s-1} \leftarrow \underbrace{D^{s_0}}_{\text{degree } 0} \leftarrow D^{s_1}.$$

- (ii.) Elements $\{g_1, \dots, g_k\} \subset D^{s_0}$ whose images modulo $\text{im}(\Omega \otimes D^{s_1})$ form a basis for

$$H^0\left(\left(\Omega \otimes_D^L \left(\frac{D^{s-n}}{\tau(P)}\right)\right)[n]\right) \simeq H^0(\Omega \otimes_D E^\bullet) \simeq \frac{\ker(\Omega \otimes_D D^{s-1} \leftarrow \Omega \otimes_D D^{s_0})}{\text{im}(\Omega \otimes_D D^{s_0} \leftarrow \Omega \otimes_D D^{s_1})}$$

5. Lift the map π_n to a chain map $\pi_\bullet : E^\bullet \rightarrow \tau(\text{Hom}_D(X^\bullet, D))$. Denote these maps $\pi_i : D^{s-i} \rightarrow D^{r-i}$.

6. Evaluate $\{\tau(\pi_0(g_1)), \dots, \tau(\pi_0(g_k))\}$ and let $\{R_1(\mathbf{x}), \dots, R_k(\mathbf{x})\}$ be their images in $(D/D \cdot \{\partial_1, \dots, \partial_n\})^{r_0} \simeq K[\mathbf{x}]^{r_0}$.

7. Return $\{R_1(\mathbf{x}), \dots, R_k(\mathbf{x})\}$, a basis for the polynomial solutions to M .

Example 3.4. Let us return to the GKZ example and apply the revised algorithm. For Step 1, we have already computed a resolution X^\bullet . Its length equals the global homological dimension. Thus for Step 2, we get a complex which is a resolution for the holonomic dual,

$$\tau(\text{Hom}(X^\bullet, D)) : 0 \leftarrow D^1 \xleftarrow{[v' \ u']} D^2 \xleftarrow{[u'-2 \ -v']} D^1 \leftarrow 0,$$

where $u' = -(\theta_1 + 2\theta_2 + 6)$ and $v' = -(\partial_1^2 + \partial_2)$. For Step 3, it follows that we get the presentation,

$$\tau(\text{Ext}_D(M, D)) \simeq \frac{D^1}{D \cdot \{u', v'\}} = \frac{D^1}{D \cdot \{\theta_1 + 2\theta_2 + 6, \partial_1^2 + \partial_2\}}$$

For Step 4, we compute the derived integration of this module. It turns out the complex $\tau(\text{Hom}(X^\bullet, D))$ is already a \tilde{V} -strict resolution when taken with the shifts $0 \leftarrow D^1[0] \leftarrow D^2[1, 0] \leftarrow D^1[1] \leftarrow 0$. The integration b -function is $s - 4$, hence according to the integration algorithm, $\Omega \otimes_D \tau(\text{Hom}(X^\bullet, D))$ is quasi-isomorphic to the finite-dimensional subcomplex,

$$0 \leftarrow \tilde{F}^4(\Omega^1[0]) \xleftarrow{[-v \ -u-1]} \tilde{F}^4(\Omega^2[-1, 0]) \xleftarrow{[-u-3 \ -v]} \tilde{F}^4(\Omega^1[-1]) \leftarrow 0.$$

(F^k and \tilde{F}^k are explained in the appendix.) Here, $\tilde{F}^4(\Omega^1[-1])$ is spanned by the 21 monomials of degree ≤ 5 ,

$$\{1, x_1, x_2, \dots, x_1^5, x_1^4 x_2, x_1^3 x_2^2, x_1^2 x_2^3, x_1 x_2^4, x_2^5\},$$

while $\tilde{F}^4(\Omega^2[0, -1])$ is spanned by the 36 monomials

$$\begin{aligned} &\{1, x_1, x_2, \dots, x_1^4, x_1^3 x_2, x_1^2 x_2^2, x_1 x_2^3, x_2^4\} \cdot \vec{e}_1 \cup \\ &\{1, x_1, x_2, \dots, x_1^5, x_1^4 x_2, x_1^3 x_2^2, x_1^2 x_2^3, x_1 x_2^4, x_2^5\} \cdot \vec{e}_2. \end{aligned}$$

The matrix $\begin{bmatrix} u' & -2 \\ v' & \end{bmatrix}$ induces a map between them whose kernel is spanned by the degree 5 polynomial $\vec{R}_1 = (x_1^5 - 20x_1^3 x_2 + 60x_1 x_2^2)$.

4. RATIONAL SOLUTIONS

A duality algorithm to compute the dimensions of $\text{Ext}_D^i(M, K[\mathbf{x}][f^{-1}])$ for holonomic M was given in [12]. In this section, we show how to extend this algorithm so as to compute an explicit basis of $\text{Hom}_D(M, K[\mathbf{x}][f^{-1}])$. The method is essentially the same as the algorithm for polynomial solutions. Also, since any rational function solution has its poles inside the singular locus of M , we obtain an algorithm to compute the rational solutions of M . Finally, we remark that a different algorithm to compute rational solutions based upon Gröbner deformations was given in [12]. Here, as otherwise, we shall use $N[f^{-1}]$ to denote $N \otimes_{K[\mathbf{x}]} K[\mathbf{x}][f^{-1}]$.

Algorithm 4.1. (Rational solutions by duality)

INPUT: $\{L_1, \dots, L_{r_1}\} \subset D^{r_0}$ such that $M = (D^{r_0}/D \cdot \{L_1, \dots, L_{r_1}\})$ is holonomic.
OUTPUT: The rational solutions $R \in K(\mathbf{x})^{r_0}$ of the system of differential equations given by $L_i \bullet R = 0$, $i = 1, \dots, r_1$.

1. Compute a polynomial f which defines the codimension 1 component of the singular locus of M (see e.g. [15]).

2. Compute a free resolution X^\bullet of M up to length $n + 1$. Let its part in cohomological degree $-n$ be denoted:

$$\dots \rightarrow D^{r-n-1} \xrightarrow{\cdot M_{-n}} D^{r-n} \xrightarrow{\cdot M_{-n+1}} D^{r-n+1} \rightarrow \dots$$

3. Form the complex $\tau(\text{Hom}_D(X^\bullet, D))$. Its part in cohomological degree n now looks like:

$$\dots \leftarrow D^{r-n-1} \xleftarrow{\cdot \tau(M_{-n})} D^{r-n} \xleftarrow{\cdot \tau(M_{-n+1})} D^{r-n+1} \leftarrow \dots$$

4. Compute a surjection

$$\varpi_n : D^{s-n} \twoheadrightarrow \ker(\cdot \tau(M_{-n})),$$

and find the preimage $\tau(P) := \varpi_n^{-1}(\text{im}(\cdot \tau(M_{-n+1})))$. Denote by $\varpi_{f,n}$ the induced map on the localizations, $D^{s-n}[f^{-1}] \twoheadrightarrow \ker(\cdot \tau(M_{-n}))[f^{-1}]$.

5. Compute the localization of $D^{s-n}/\tau(P)$ at f using the algorithm of [13]. This produces a presentation,

$$\bar{\varphi} : \frac{D^{s-n}}{\tau(Q)} \xrightarrow{\simeq} \left(\frac{D^{s-n}[f^{-1}]}{\tau(P)[f^{-1}]} \right)$$

$$e_i \bmod \tau(Q) \longrightarrow (e_i \bmod \tau(P)) \otimes f^{-a_i}$$

6. Compute the derived integration module $H^0((\Omega \otimes_D^L D^{s-n}/\tau(Q))[n])$ using Algorithm 8.8. In particular, this algorithm produces

- (i.) A \tilde{V} -strict free resolution E^\bullet of $D^{s-n}/\tau(Q)$ of length $n + 1$,

$$E^\bullet : 0 \leftarrow \underbrace{D^{s-n}}_{\text{degree } n} \leftarrow D^{s-n+1} \leftarrow \dots \leftarrow D^{s-1} \leftarrow \underbrace{D^{s_0}}_{\text{degree } 0} \leftarrow D^{s_1}.$$

- (ii.) Elements $\{g_1, \dots, g_k\} \subset D^{s_0}$ whose images form a basis for

$$H^0\left(\left(\Omega \otimes_D^L \frac{D^{s-n}}{\tau(Q)}\right)[n]\right) \simeq H^0(\Omega \otimes_D E^\bullet) \simeq \frac{\ker(\Omega \otimes_D D^{s-1} \leftarrow \Omega \otimes_D D^{s_0})}{\text{im}(\Omega \otimes_D D^{s_0} \leftarrow \Omega \otimes_D D^{s_1})}$$

7. Let π_n be the composition

$$\varpi_{f,n} \circ \varphi : D^{s-n} \rightarrow D^{s-n}[f^{-1}] \rightarrow \ker(\cdot \tau(M_{-n}))[f^{-1}],$$

where $\varphi : D^{s-n} \rightarrow D^{s-n}[f^{-1}]$ is the map defined by $e_i \mapsto e_i \otimes f^{-a_i}$. Lift π_n to a chain map $\pi_\bullet : E^\bullet \rightarrow \text{Hom}_D(X^\bullet, D)[f^{-1}]$.

8. Evaluate $\{\tau(\pi_0(g_1)), \dots, \tau(\pi_0(g_k))\} \subset D^{r_0}[f^{-1}]$ and let $\{R_1(\mathbf{x}), \dots, R_k(\mathbf{x})\}$ be their images in $(D[f^{-1}]/D[f^{-1}] \cdot \{\partial_1, \dots, \partial_n\})^{r_0} \simeq K[\mathbf{x}][f^{-1}]^{r_0}$.
9. Return $\{R_1(\mathbf{x}), \dots, R_k(\mathbf{x})\}$, a basis for the rational solutions to M .

Proof. As explained in [12], any rational solution of M has its poles contained inside the singular locus of M . The proof is now essentially the same as for the polynomial case. The space of rational solutions can be identified with the 0-th cohomology of the complex (6), which specializes to

$$\begin{aligned} \text{Hom}_D(X^\bullet, D) \otimes_D K[\mathbf{x}][f^{-1}] &\simeq \text{Hom}_D(X^\bullet, D)[f^{-1}] \otimes_{D[f^{-1}]} K[\mathbf{x}][f^{-1}] \\ &\simeq \text{Hom}_D(X^\bullet, D)[f^{-1}] \otimes_{D[f^{-1}]} D[f^{-1}] \otimes_D K[\mathbf{x}] \\ &\simeq \text{Hom}_D(X^\bullet, D)[f^{-1}] \otimes_D K[\mathbf{x}] \end{aligned}$$

Since the complex $\text{Hom}_D(X^\bullet, D)$ is exact except in cohomological degree n where its cohomology is $\text{Ext}_D^n(M, D)$, and since localization is exact, the complex $\text{Hom}_D(X^\bullet, D)[f^{-1}]$ remains exact except in cohomological degree n where its cohomology becomes $\text{Ext}_D^n(M, D)[f^{-1}]$. Hence $\text{Hom}_D(X^\bullet, D)[f^{-1}] \otimes_D K[\mathbf{x}]$ computes the derived integration modules of $\text{Ext}_D^n(M, D)[f^{-1}]$ in the category

of right D -modules. The above algorithm computes cohomology classes for the derived integration modules and transfers them back to cohomology classes of $\mathrm{Hom}_D(X^\bullet, D) \otimes_D K[\mathbf{x}][f^{-1}]$. \square

Remark 4.2. Let us explain how the lifting of π_n to a chain map in Step 7 may be accomplished algorithmically. We wish to do computations in terms of D and not $D[f^{-1}]$. The idea is that localization is exact, hence any boundary in $\mathrm{Hom}_D(X^\bullet, D)[f^{-1}]$ is the localization of a boundary in $\mathrm{Hom}_D(X^\bullet, D)$. Suppose we have computed $\pi_j : D^{s-j} \rightarrow D^{r-j}[f^{-1}]$. Then to compute π_{j-1} , we first compute the images ℓ_i of $e_i \in D^{s_{j+1}}$ under $D^{s_{j+1}} \rightarrow D^{s-j} \xrightarrow{\pi_j} D^{r-j}[f^{-1}]$. Because the existing π_n, \dots, π_j are the beginning of a chain map, the ℓ_i are in the image of $D^{r-j+1}[f^{-1}] \rightarrow D^{r-j}[f^{-1}]$. Now we use the fact that localization is exact, which means for sufficiently large m_i , $f^{m_i}\ell_i$ is in the image of $D^{r-j+1} \rightarrow D^{r-j}$. To find valid m_i , we can multiply ℓ_i by successively higher powers of f and test for membership at each step via Gröbner basis over D . Now compute any preimage P_i of $f^{m_i}\ell_i$ in D^{r-j} . The map $\pi_{j-1} : D^{s_{j+1}} \rightarrow D^{r-j+1}[f^{-1}]$ may be defined by sending $e_i \mapsto \frac{1}{f^{m_i}}P_i$.

Example 4.3. The following system of differential equations of two variables is called the Appell differential equation $F_1(a, b, b', c)$:

$$\begin{aligned} &\theta_x(\theta_x + \theta_y + c - 1) - x(\theta_x + \theta_y + a)(\theta_x + b), \\ &\theta_y(\theta_x + \theta_y + c - 1) - y(\theta_x + \theta_y + a)(\theta_y + b'), \\ &(x - y)\partial_x\partial_y - b'\partial_x + b\partial_y, \end{aligned}$$

where a, b, b', c are complex parameters. In [12], the dimension of the rational solution space of $F_1(2, -3, -2, 5)$ was computed using the duality method. This system has rank 3, and its solution space is spanned by a polynomial, a rational solution with pole along x , and a rational solution with pole along y .

Let us obtain the solution with pole along x explicitly. In [12] was computed a resolution for $F_1(2, -3, -2, 5)$,

$$X^\bullet : 0 \rightarrow D^1 \xrightarrow{\cdot M_{-1}} D^2 \xrightarrow{\cdot M_0} D^1 \rightarrow 0,$$

so that $\tau(\mathrm{Hom}_D(X^\bullet, D))$ is a resolution for $\tau(\mathrm{Ext}_D^n(M, D)) = F_1(-1, 4, 2, -3)$,

$$\tau(\mathrm{Hom}_D(X^\bullet, D)) : 0 \leftarrow D^1 \xleftarrow{\cdot \tau(M_{-1})} D^2 \xleftarrow{\cdot \tau(M_0)} D^1 \leftarrow 0.$$

where,

$$\begin{aligned} \tau(M_{-1}) &= \begin{bmatrix} (\theta_x + 4)\partial_y - (\theta_y + 2)\partial_x \\ (y^2 - y)(\partial_x\partial_y + \partial_y^2) + 2(x + y)\partial_x - 2y\partial_y - 2\partial_x + 7\partial_y - 2 \end{bmatrix} \\ \tau(M_0) &= \begin{bmatrix} (y^2 - y)(\partial_x\partial_y + \partial_y^2) + 2(x + 2y)\partial_x - 3\partial_x + 6\partial_y - 4 \\ -(\theta_x + 4)\partial_y + (\theta_y + 3)\partial_x \end{bmatrix}^T. \end{aligned}$$

It was also computed that the localization has presentation

$$\begin{aligned} \bar{\varphi} : \frac{D}{\tau(Q)} &\xrightarrow{\simeq} \left(\frac{D[x^{-1}]}{\mathrm{im}(\cdot \tau(M_{-1}))[x^{-1}]} \right) \simeq \tau(\mathrm{Ext}_D^n(M, D))[x^{-1}] \\ 1 \bmod \tau(Q) &\longrightarrow (1 \bmod \mathrm{im}(\cdot \tau(M_{-1}))) \otimes x^{-7} \end{aligned}$$

where

$$\tau(Q) = D \cdot \left\{ \begin{array}{l} (\theta_x \theta_y + \theta_y^2 + 8\theta_y + 2\theta_x + 12) - (\theta_x + \theta_y + 4)\partial_y \\ (\theta_x \theta_y + 2\theta_x + 7\theta_y + 14) - (\theta_x + 10)x\partial_y \end{array} \right\},$$

and that $D/\tau(Q)$ has a \tilde{V} -strict resolution,

$$E^\bullet : 0 \longleftarrow D^1[0] \xleftarrow{\begin{bmatrix} u_1 \\ u_2 \end{bmatrix}} D^2[0, -1] \xleftarrow{[v_1, v_2]} D^1[-1] \longleftarrow 0,$$

where

$$\begin{aligned} u_1 &= x^2 \partial_x \partial_y - xy \partial_x \partial_y - 2x \partial_x + 11x \partial_y - 7y \partial_y - 14 \\ u_2 &= x^3 \partial_x^2 + x^3 \partial_x \partial_y - x^2 \partial_x^2 - x^2 \partial_x \partial_y + 16x^2 \partial_x + 11x^2 \partial_y + \\ &\quad 4xy \partial_y - 9x \partial_x - 11x \partial_y + 52x - 7 \\ v_1 &= x^3 \partial_x^2 + x^3 \partial_x \partial_y - x^2 \partial_x^2 - x^2 \partial_x \partial_y + 16x^2 \partial_x + 12x^2 \partial_y + \\ &\quad 4xy \partial_y - 8x \partial_x - 11x \partial_y + 52x - 6 \\ v_2 &= -x^2 \partial_x \partial_y + xy \partial_x \partial_y + 2x \partial_x - 11x \partial_y + 6y \partial_y + 12. \end{aligned}$$

We would like to construct a chain map $\pi_\bullet : E^\bullet \rightarrow \tau(\text{Hom}_D(X^\bullet, D))$ which lifts the map $\pi_2 : D^1[0] \rightarrow D[x^{-1}]^1$ defined by $1 \mapsto x^{-7}$. To compute the next map $\pi_1 : D^2[0, -1] \rightarrow D[x^{-1}]^2$, we need to find preimages of the elements $\pi_2 \circ (\cdot[u_1, u_2]^T)(e_1)$ and $\pi_2 \circ (\cdot[u_1, u_2]^T)(e_2)$ under the map $(\cdot\tau(M_{-1})) : D^1[x^{-1}] \leftarrow D^2[x^{-1}]$. Note that

$$\begin{aligned} \pi_2 \circ (\cdot[u_1, u_2]^T)(e_1) &= u_1 \cdot x^{-7} \\ &= x^{-6}((\theta_x + 4)\partial_y - (\theta_y + 2)\partial_x) \end{aligned}$$

It follows that $\pi_2 \circ (\cdot[u_1, u_2]^T)(x^6 e_1) = (\cdot\tau(M_{-1}))(e'_1)$ so that we may set $\pi_1(e_1) = x^{-6}e'_1$. In a similar manner, we obtain the chain map,

$$\begin{array}{ccccccc} E^\bullet : 0 & \longleftarrow & D^1[0] & \xleftarrow{\begin{bmatrix} u_1 \\ u_2 \end{bmatrix}} & D^2[0, -1] & \xleftarrow{[v_1, v_2]} & D^1[-1] \longleftarrow 0 \\ & & \downarrow \pi_2 \cdot [x^{-7}] & & \downarrow \pi_1 \cdot \begin{bmatrix} x^{-6} & 0 \\ x^{-6}a & x^{-6}b \end{bmatrix} & & \downarrow \pi_0 \cdot [x^{-5}c] \\ \tau(\text{Hom}_D(X^\bullet, D)) : 0 & \longleftarrow & D^1[x^{-1}] & \xleftarrow{\cdot\tau(M_{-1})} & D^2[x^{-1}] & \xleftarrow{\cdot\tau(M_0)} & D^1[x^{-1}] \longleftarrow 0, \end{array}$$

where

$$\begin{aligned} a &= -\frac{1}{2}(y^2 - y)(\partial_x + \partial_y) + x + 2y - \frac{9}{2} \\ b &= \frac{1}{2}(x - y)\partial_x + 2 \\ c &= \frac{1}{2}(x - y)\partial_x + \frac{5}{2} \end{aligned}$$

The integration b -function is $(s-11)(s-4)(s-1)$, hence according to the integration algorithm, $\Omega \otimes_D E^\bullet$ is quasi-isomorphic to its subcomplex $\tilde{F}^{11}(\Omega \otimes_D E^\bullet)$. Using Macaulay 2, we find that $\ker(\Omega \xleftarrow{[v_1, v_2]} \Omega^2)$ is 2-dimensional and spanned by,

$$\begin{aligned} g_1 &= x^9 y - \frac{3}{2}x^9 - 6x^8 y + \frac{48}{5}x^8 + \frac{72}{5}x^7 y \\ &\quad - \frac{126}{5}x^7 - \frac{84}{5}x^6 y + \frac{168}{5}x^6 + \frac{42}{5}x^5 y - 21x^5 \\ g_2 &= -x^3 + \frac{2}{7}x^2 y + \frac{4}{7}x^2 - \frac{3}{14}xy - \frac{3}{28}x + \frac{1}{21}y \end{aligned}$$

The residue class of $\tau(\pi_0(g_1))$ yields the polynomial solution,

$$\begin{aligned} &(2y^2 - 6y + \frac{24}{5})x^3 + (-9y^2 + \frac{144}{5}y - \frac{165}{5})x^2 \\ &+ (\frac{72}{5}y^2 - \frac{252}{5}y + \frac{252}{5})x + (-\frac{42}{5}y^2 + \frac{168}{5}y - 42), \end{aligned}$$

while the residue class of $\tau(\pi_0(g_2))$ yields the rational solution,

$$\left(-6x^4 + 4x^3y - \frac{6}{7}x^2y^2 + 4x^3 - \frac{24}{7}x^2y + \frac{6}{7}xy^2 - \frac{6}{7}x^2 + \frac{6}{7}xy - \frac{5}{21}y^2\right)x^{-6}$$

By similar methods, we obtain the rational solutions with pole along y ,

$$\left(\begin{array}{l} x^3y^2 - \frac{21}{4}x^2y^3 + \frac{21}{2}xy^4 - \frac{35}{4}y^5 - \frac{5}{4}x^3y + 6x^2y^2 \\ -\frac{21}{2}xy^3 + 7y^4 + \frac{5}{12}x^3 - \frac{15}{8}x^2y + 3xy^2 - \frac{7}{4}y^3 \end{array}\right)y^{-7}.$$

Together these solutions span the holomorphic solution space in a neighborhood of any point away from $xy = 0$.

Remark 4.4. In the next section, we give an algorithm to compute $\text{Hom}_D(M, N)$ for arbitrary holonomic M and N . Using it with $N = K[\mathbf{x}][f^{-1}]$, we get a similar but computationally different duality method to compute rational solutions. The basic difference is that the algorithm of this section uses computations over D and in principle over $D[f^{-1}]$, while the algorithm of the next section uses computations over D_{2n} , the Weyl algebra in twice as many variables. From the computational perspective, we believe the algorithm of this section is more efficient.

5. HOLONOMIC SOLUTIONS

In this section, we give an algorithm to compute a basis of $\text{Hom}_D(M, N)$ for holonomic left D -modules M and N . We will use the following notation. As before, D will denote the ring of differential operators in the variables x_1, \dots, x_n with derivations $\partial_1, \dots, \partial_n$. Occasionally we will also write D_n or D_x for D . In a similar fashion, D_y will stand for the ring of differential operators in the variables y_1, \dots, y_n with derivations $\delta_1, \dots, \delta_n$.

If X is a D_x -module and Y a D_y -module then we denote by $X \boxtimes Y$ the external product of X and Y . It equals the tensor product of X and Y over the field K , equipped with its natural structure as a module over $D_{2n} = D_x \boxtimes D_y$, the ring of differential operators in $x_1, \dots, x_n, y_1, \dots, y_n$ with derivations $\{\partial_i, \delta_j\}_{1 \leq i, j \leq n}$. In addition, let η denote the algebra isomorphism,

$$\eta : D_{2n} \longrightarrow D_{2n} \quad \left\{ \begin{array}{ll} x_i \mapsto \frac{1}{2}x_i - \delta_i, & \partial_i \mapsto \frac{1}{2}y_i + \partial_i, \\ y_i \mapsto -\frac{1}{2}x_i - \delta_i, & \delta_i \mapsto \frac{1}{2}y_i - \partial_i \end{array} \right\}_{i=1}^n,$$

and let Δ and Λ denote the right D_{2n} -modules,

$$\Delta := \frac{D_{2n}}{\{x_i - y_i, \partial_i + \delta_i : 1 \leq i \leq n\} \cdot D_{2n}} \quad \Lambda := \frac{D_{2n}}{\mathbf{x}D_{2n} + \mathbf{y}D_{2n}} = \eta(\Delta).$$

As mentioned in the introduction, an algorithm to compute the dimensions of $\text{Ext}_D^i(M, N)$ was given in [12] based upon the isomorphisms (1) and (2):

$$\begin{aligned} \text{Ext}_D^i(M, N) &\cong \text{Tor}_{n-i}^D(\text{Ext}_D^n(M, D), N) \\ \text{Tor}_j^D(M', N) &\simeq \text{Tor}_j^{D_{2n}}(D_{2n}/\{x_i - y_i, \partial_i + \delta_i\}_{i=1}^n \cdot D_{2n}, \tau(M') \boxtimes N). \end{aligned}$$

Combining these isomorphisms where $M' = \text{Ext}_D^n(M, D)$ produces

(10)

$$\text{Ext}_D^i(M, N) \simeq \text{Tor}_j^{D_{2n}}(D_{2n}/\{x_i - y_i, \partial_i + \delta_i\}_{i=1}^n \cdot D_{2n}, \tau(\text{Ext}_D^n(M, D)) \boxtimes N)$$

In order to compute $\text{Hom}_D(M, N)$ explicitly, we will trace the isomorphism (10). We explain how to do this step by step in the following algorithm. The motivation behind the algorithm is discussed in the proof.

Algorithm 5.1. (Holonomic solutions by duality)

INPUT: Presentations $M = D^{r_0}/M_0$ and $N = D^{s_0}/N_0$ of holonomic left D -modules.

OUTPUT: A basis for $\text{Hom}_D(M, N)$.

1. Compute finite free resolutions X^\bullet and Y^\bullet of M and N ,

$$X^\bullet : 0 \rightarrow \underbrace{D^{r-a}}_{\text{degree } -a} \xrightarrow{\cdot M_{-a+1}} \dots \rightarrow D^{r-1} \xrightarrow{\cdot M_0} \underbrace{D^{r_0}}_{\text{degree } 0} \rightarrow M \rightarrow 0$$

$$Y^\bullet : 0 \rightarrow \underbrace{D^{s-b}}_{\text{degree } -b} \xrightarrow{\cdot N_{-b+1}} \dots \rightarrow D^{s-1} \xrightarrow{\cdot N_0} \underbrace{D^{s_0}}_{\text{degree } 0} \rightarrow N \rightarrow 0$$

Also, dualize X^\bullet and apply the standard transposition to obtain,

$$\tau(\text{Hom}_D(X^\bullet, D)) : 0 \leftarrow \underbrace{D^{r-a}}_{\text{degree } a} \xleftarrow{\cdot \tau(M_{-a+1})} \dots \leftarrow D^{r-1} \xleftarrow{\cdot \tau(M_0)} \underbrace{D^{r_0}}_{\text{degree } 0} \leftarrow 0.$$

2. Form the double complex $\tau(\text{Hom}_D(X^\bullet, D)) \boxtimes Y^\bullet$ of left D_{2n} -modules and its total complex

$$Z^\bullet : 0 \leftarrow \underbrace{D_{2n}^{t_a}}_{\text{degree } a} \leftarrow \dots \leftarrow \underbrace{D_{2n}^{t_0}}_{\text{degree } 0} \leftarrow \dots \leftarrow D_{2n}^{t_{-b}} \leftarrow 0$$

where

$$D_{2n}^{t_k} = \bigoplus_{i-j=k} D^{r-i} \boxtimes D^{s-j}.$$

Let the part of Z^\bullet in cohomological degree n be denoted,

$$D_{2n}^{t_{n+1}} \xleftarrow{\cdot T_n} D_{2n}^{t_n} \xleftarrow{\cdot T_{n-1}} D_{2n}^{t_{n-1}}$$

3. Compute a surjection $\pi_n : D_{2n}^{u_n} \twoheadrightarrow \ker(\cdot \eta(T_n))$, and find the preimage $P := \pi_n^{-1}(\text{im}(\cdot \eta(T_{n-1})))$.
4. Compute the derived restriction module $H^0((\Lambda \otimes_{D_{2n}}^L D_{2n}^{u_n}/P)[n])$ using Algorithm 8.6. In particular, this algorithm produces,
 - (i.) A V -strict free resolution E^\bullet of D^{s_n}/P of length $n+1$,

$$E^\bullet : 0 \leftarrow \underbrace{D_{2n}^{u_n}}_{\text{degree } n} \leftarrow D_{2n}^{u_{n-1}} \leftarrow \dots \leftarrow D_{2n}^{u_1} \leftarrow \underbrace{D_{2n}^{u_0}}_{\text{degree } 0} \leftarrow D_{2n}^{u_{-1}}.$$

- (ii.) Elements $\{g_1, \dots, g_k\} \subset D_{2n}^{u_0}$ whose images in $\Lambda \otimes_{D_{2n}} E^\bullet$ form a basis for

$$H^0\left(\left(\Lambda \otimes_{D_{2n}}^L \frac{D_{2n}^{u_n}}{P}\right)[n]\right) \simeq H^0(\Lambda \otimes_{D_{2n}} E^\bullet) \simeq \frac{\ker(\Lambda \otimes_{D_{2n}} D_{2n}^{u_1} \leftarrow \Lambda \otimes_{D_{2n}} D_{2n}^{u_0})}{\text{im}(\Lambda \otimes_{D_{2n}} D_{2n}^{u_0} \leftarrow \Lambda \otimes_{D_{2n}} D_{2n}^{u_{-1}})}$$

5. Lift the map π_n to a chain map $\pi_\bullet : E^\bullet \rightarrow \eta(Z^\bullet)$. Denote these maps by $\pi_i : D^{u_i} \rightarrow D^{r_i}$.
6. Compute the image of each g_i under the composition of chain maps,

$$\begin{array}{ccc} E^\bullet & \Delta \otimes_{D_{2n}} Z^\bullet \xrightarrow{\simeq} \text{Tot}^\bullet(\text{Hom}_D(X^\bullet, D) \otimes_D Y^\bullet) & \\ \pi_\bullet \downarrow & \uparrow & \downarrow p_1 \\ \eta(Z^\bullet) & \xrightarrow{\eta^{-1}} Z^\bullet & \text{Hom}_D(X^\bullet, N) \end{array}$$

Here p_1 is the projection onto $\text{Hom}_D(X^\bullet, D) \otimes Y^0$ followed by factorization through N_0 . These are all chain maps of complexes of vector spaces. Step by step, we do the following. Evaluate $\{L_1 = \eta^{-1}(\pi_0(g_1)), \dots, L_k = \eta^{-1}(\pi_0(g_k))\}$, and write each L_i in terms of the decomposition,

$$L_i = \bigoplus_j L_{i,j} \in \bigoplus_j D^{r-j} \boxtimes D^{s-j} \quad (= D_{2n}^{t_0}).$$

Now re-express $L_{i,0}$ modulo $\{x_i - y_i, \partial_i + \delta_i : 1 \leq i \leq n\} \cdot D_{2n} \otimes_{D_{2n}} (D^{r_0} \boxtimes D^{s_0})$ so that x_i and ∂_j do not appear in any component. Using the identification $D^{r_0} \boxtimes D^{s_0} \simeq D_{2n}^{s_0} e_1 \oplus \dots \oplus D_{2n}^{s_0} e_r$, where $\{e_i\}$ forms the canonical D -basis for D^{r_0} , we then get an expression

$$L_{i,0} = \ell_{i,1} e_1 + \dots + \ell_{i,r_0} e_{r_0} \in (D_y)^{s_0} e_1 \oplus \dots \oplus (D_y)^{s_0} e_{r_0}.$$

Let $\{\bar{\ell}_{i,1}, \dots, \bar{\ell}_{i,r_0}\}$ be the images in $(D^{s_0}/N_0) \simeq N$. Finally, set $\phi_i \in \text{Hom}_D(M, N)$ to be the map induced by

$$\{e_1 \mapsto \bar{\ell}_{i,1}, e_2 \mapsto \bar{\ell}_{i,2}, \dots, e_{r_0} \mapsto \bar{\ell}_{i,r_0}\}.$$

7. Return $\{\phi_1, \dots, \phi_k\}$, a basis for $\text{Hom}_D(M, N)$.

Proof. The main idea behind the algorithm is to adapt the proof of Theorem 2.1. In that proof, we saw that $\text{Tot}^\bullet(\text{Hom}(X^\bullet, D) \otimes_D Y^\bullet) \xrightarrow{p_1} \text{Hom}_D(X^\bullet, N)$ is a quasi-isomorphism. Thus it suffices to compute explicit generating classes for

$$H^0(\text{Tot}^\bullet(\text{Hom}_D(X^\bullet, D) \otimes_D Y^\bullet)) \xrightarrow{\sim} H^0(\text{Hom}_D(X^\bullet, N)) \simeq \text{Hom}_D(M, N).$$

Here, the double complex $\text{Hom}_D(X^\bullet, D) \otimes_D Y^\bullet$ is in some sense easier to digest because it consists entirely of free D -modules. However, it too only carries the structure of a complex of infinite-dimensional vector spaces, making its cohomology no easier to compute than the cohomology of $\text{Hom}_D(X^\bullet, N)$.

Thus, we instead are led to consider the double complex $\tau(\text{Hom}_D(X^\bullet, D)) \boxtimes Y^\bullet$ of Step 2, whose total complex T^\bullet does carry the structure of a complex of left D_{2n} -modules. Moreover, we can get back to the original double complex by “restricting back to the diagonal”. In other words, we claim that as a double complex of vector spaces, $\text{Hom}_D(X^\bullet, D) \otimes_D Y^\bullet$ can be naturally identified with the double complex,

$$\Delta \otimes_D (\tau(\text{Hom}_D(X^\bullet, D)) \boxtimes Y^\bullet).$$

To make the identification, first note that the natural map

$$D_y \longrightarrow \frac{D_{2n}}{\{x_i - y_i, \partial_i + \delta_i : 1 \leq i \leq n\} \cdot D_{2n}} = \Delta$$

is an isomorphism of left D_y -modules. Let $\{e_1, \dots, e_r\}$ denote the canonical basis of a free module D^r . Then an arbitrary element of $\Delta \otimes_{D_{2n}} (D_x^r \boxtimes D_y^s)$ can be expressed uniquely as $\sum_k e_k \boxtimes m_k$, where $m_k \in D_y^s$. Similarly, an element of $D^r \otimes_D D^s$ can be expressed uniquely as $\sum_k e_k \otimes m_k$ where $m_k \in D^s$. Hence we get an isomorphic identification as D_n -modules of $\Delta \otimes_{D_{2n}} (D_x^r \boxtimes D_y^s)$ and $D^r \otimes_D D^s$. In particular, this shows that the modules appearing in the double complexes are the same.

It remains to show that the maps in the double complexes can also be identified. An arbitrary vertical map of $\Delta \otimes_{D_{2n}} (\tau(\text{Hom}_D(X^\bullet, D)) \boxtimes Y^\bullet)$ acts on an arbitrary

element $\sum_k 1 \otimes e_k \boxtimes m_k$ according to,

$$\begin{array}{ccc} \Delta \otimes_{D_{2n}} (D_x^{r_i} \boxtimes D_y^{s_j}) & \xrightarrow{\sum_i (-1)^i e_k \boxtimes (\cdot N_j)(m_k)} & \Delta \otimes_{D_{2n}} (D_x^{r_i} \boxtimes D_y^{s_j}) \\ \uparrow \text{id}_\Delta \otimes (-\text{id}_{r_i})^i \boxtimes (\cdot N_j) & & \uparrow \sum_k 1 \otimes e_k \boxtimes m_k \\ \Delta \otimes_{D_{2n}} (D_x^{r_i} \boxtimes D_y^{s_{j+1}}) & & \Delta \otimes_{D_{2n}} (D_x^{r_i} \boxtimes D_y^{s_{j+1}}) \end{array}$$

This is exactly the way the corresponding vertical map in $\text{Hom}_D(X^\bullet, D) \otimes_D Y^\bullet$ works on the corresponding element:

$$\begin{array}{ccc} D_x^{r_i} \otimes_D D_y^{s_j} & \xrightarrow{\sum_k (-1)^i e_k \otimes (\cdot N_j)(m_k)} & D_x^{r_i} \otimes_D D_y^{s_j} \\ \uparrow (-\text{id}_{r_i})^i \otimes (\cdot N_j) & & \uparrow \sum_k e_k \otimes m_k \\ D_x^{r_i} \otimes_D D_y^{s_{j+1}} & & D_x^{r_i} \otimes_D D_y^{s_{j+1}} \end{array}$$

Likewise, an arbitrary horizontal map of $\Delta \otimes_{D_{2n}} (\tau(\text{Hom}_D(X^\bullet, D)) \boxtimes Y^\bullet)$ acts on an arbitrary element according to,

$$\begin{array}{ccc} \Delta \otimes_{D_{2n}} (D_x^{r_{i+1}} \boxtimes D_y^{s_j}) & \xrightarrow{\text{id}_\Delta \otimes (\cdot \tau(M_i)) \boxtimes 1} & \Delta \otimes_{D_{2n}} (D_x^{r_i} \boxtimes D_y^{s_j}) \\ \sum_k 1 \otimes e_k \boxtimes m_k & \longrightarrow & \sum_k 1 \otimes (\cdot \tau(M_i))(e_k) \boxtimes m_k \end{array}$$

Here, we would like to re-express the image $\sum_k 1 \otimes (\cdot \tau(M_i))(e_k) \boxtimes m_k$ in the form $\sum_k 1 \otimes e_k \boxtimes n_k$. To help us, note the following computation in $\Delta \otimes_{D_{2n}} (D_x^r \boxtimes D_y^s)$:

$$(1 \otimes x^\alpha \partial^\beta e_i \boxtimes m) = 1 \otimes \partial^\beta e_i \boxtimes y^\alpha m = 1 \otimes e_i \boxtimes (-\delta)^\beta y^\alpha m = 1 \otimes e_i \boxtimes \tau(y^\alpha \delta^\beta) m.$$

Using it, we get that

$$\begin{aligned} \sum_k 1 \otimes (\cdot \tau(M_i))(e_k) \boxtimes m_k &= \sum_k \sum_j 1 \otimes \tau(M_i)_{jk} e_j \boxtimes m_k \\ &= \sum_k \sum_j 1 \otimes e_j \boxtimes \tau(\tau(M_i)_{jk}) m_k \\ &= \sum_k \sum_j 1 \otimes e_j \boxtimes (M_i)_{jk} m_k \end{aligned}$$

This is exactly the way the corresponding horizontal map in $\text{Hom}_D(X^\bullet, D) \otimes_D Y^\bullet$ works on an arbitrary element:

$$\begin{array}{ccc} D^{r_{i+1}} \otimes_D D^{s_j} & \xrightarrow{(M_i \cdot) \otimes \text{id}_{s_j}} & D^{r_i} \otimes_D D^{s_j} \\ \sum_k e_k \otimes m_k & \longrightarrow & \sum_k \sum_j e_j \otimes (M_i)_{jk} m_k \end{array}$$

Thus, we have given an explicit identification of $\Delta \otimes_D (\tau(\text{Hom}_D(X^\bullet, D)) \boxtimes Y^\bullet)$ and $\text{Hom}_D(X^\bullet, D) \otimes_D Y^\bullet$.

The task now becomes to compute explicit cohomology classes which are a basis for $H^0(\Delta \otimes_{D_{2n}} Z^\bullet)$. To do this, we note that Z^\bullet is exact except in cohomological degree n , where its cohomology is $\tau(\text{Ext}_D^n(M, D)) \boxtimes N$. This follows because $\tau(\text{Hom}_D(X^\bullet, D))$ is exact by holonomicity except in degree n , where its cohomology is $\tau(\text{Ext}_D^n(M, D))$, and Y^\bullet is exact except in degree 0, where its cohomology is

N . In other words, the complex $\Delta \otimes_{D_{2n}} Z^\bullet$ is in some sense a restriction complex. Namely, after applying the algebra isomorphism η , we get an honest restriction complex $\Lambda \otimes \eta(Z^\bullet)$ for the restriction of $\eta(\tau(\text{Ext}_D^n(M, D)) \boxtimes N)$ to the origin (the restriction complex of a left D_{2n} -module M' is by definition $\Lambda \otimes_{D_{2n}}^L M'$).

We can thus compute the cohomology groups of $\Lambda \otimes_{D_{2n}} \eta(Z^\bullet)$ by applying the restriction algorithm. However, since we are after explicit representatives for the cohomology classes, we need to use a presentation of $\eta(\tau(\text{Ext}_D^n(M, D)) \boxtimes N)$ which is compatible with $\eta(Z^\bullet)$. This is the content of Step 3. Once equipped with a compatible presentation, we apply the restriction algorithm to it, which is the content of Step 4. This step produces explicit cohomology classes of $\Lambda \otimes_{D_{2n}} E^\bullet$, where E^\bullet is a V -strict resolution of $\eta(\tau(\text{Ext}_D^n(M, D)) \boxtimes N)$. To then get explicit cohomology classes of $\Lambda \otimes_{D_{2n}} \eta(Z^\bullet)$, we construct a chain map between E^\bullet and $\eta(Z^\bullet)$, which is the content of Step 5. The cohomology classes can now be transported to $\Lambda \otimes_{D_{2n}} \eta(Z^\bullet)$ using the chain map, then to $\Delta \otimes_{D_{2n}} Z^\bullet$ using η^{-1} , then to $\text{Tot}^\bullet(\text{Hom}_D(X^\bullet, D) \otimes_D Y^\bullet)$ using the natural identification described earlier, and finally to the complex $\text{Hom}_D(X^\bullet, N)$ using the natural augmentation map. These steps are all grouped together in Step 6. This completes the proof of the algorithm. \square

Remark 5.2. In Algorithms 3.3 and 4.1 we used the integration algorithm as the main workhorse, while in 5.1 we used restriction. As should become apparent from the appendix, these are really mirror images of each other. Rational and polynomial solutions naturally fit into the integration picture. On the other hand, most papers are written in the language of restriction.

Example 5.3. Let $M = D/D \cdot (\partial - 1)$ and $N = D/D \cdot (\partial - 1)^2$, where D is the first Weyl algebra. Then for Step 1, we have the resolutions,

$$X^\bullet : 0 \rightarrow D^1 \xrightarrow{\cdot(\partial-1)} D^1 \rightarrow 0 \quad Y^\bullet : 0 \rightarrow D^1 \xrightarrow{\cdot(\partial-1)^2} D^1 \rightarrow 0$$

For Step 2, we form the complex $Z^\bullet = \text{Tot}(\tau(\text{Hom}_D(X^\bullet, D)) \boxtimes Y^\bullet)$,

$$Z^\bullet : 0 \leftarrow \underbrace{D_2^1}_{\text{degree 1}} \xleftarrow{\cdot \begin{bmatrix} (\partial_x+1) \\ (\partial_y-1)^2 \end{bmatrix}} \underbrace{D_2^2}_{\text{degree -1}} \xleftarrow{\cdot [(\partial_y-1)^2, -(\partial_x+1)]} \underbrace{D_2^1}_{\text{degree 0}} \leftarrow 0$$

For Steps 3-5, we get the output,

$$\begin{array}{ccccccc} \eta(Z^\bullet) : 0 \leftarrow & D_2^1 & \xleftarrow{\cdot \begin{bmatrix} \frac{1}{2}y + \partial_x + 1 \\ (\frac{1}{2}y - \partial_x - 1)^2 \end{bmatrix}} & D_2^2 & \xleftarrow{\cdot [(\frac{1}{2}y - \partial_x - 1)^2, -\frac{1}{2}y - \partial_x - 1]} & D_2^1 & \leftarrow 0 \\ & \uparrow \pi_1 = \cdot [1] & & \uparrow \pi_0 = \begin{bmatrix} \frac{3}{2}y - \partial_x - 1 & 0 \\ 0 & 1 \end{bmatrix} & & & \\ E^\bullet : 0 \leftarrow & D_2^1[0] & \xleftarrow{\cdot \begin{bmatrix} \frac{1}{2}y + \partial_x + 1 \\ y^2 \end{bmatrix}} & D_2^2[-1, 2] & \xleftarrow{\cdot [y^2, -\frac{1}{2}y - \partial_x - 1]} & D_2^1[1] & \leftarrow 0 \end{array}$$

The complex E^\bullet is a V -strict resolution of the cohomology of $\eta(Z^\bullet)$ at degree 1, and the restriction b -function is $b(s) = (s+1)(s+2)$. Hence $\Lambda \otimes_D E^\bullet$ is quasi-isomorphic to its sub-complex $F^{-1}(\Lambda \otimes_D E^\bullet)$

$$0 \leftarrow 0 \xleftarrow{\cdot \begin{bmatrix} \frac{1}{2}y + \partial_x + 1 \\ y^2 \end{bmatrix}} \text{Span}_K \left\{ \begin{array}{c} 0 \oplus \overline{1} \\ 0 \oplus \overline{\partial_x} \\ 0 \oplus \overline{\partial_y} \end{array} \right\} \xleftarrow{\cdot [y^2, -\frac{1}{2}y - \partial_x - 1]} \text{Span}_K \{\overline{1}\} \leftarrow 0$$

Hence the cohomology $H^0(\Lambda \otimes_D E^\bullet)$ is spanned by $\{0 \oplus \bar{1}, 0 \oplus \bar{\partial}_y\}$. Applying π_0 , $H^0(\Lambda \otimes_D \eta(Z^\bullet))$ is spanned by the images of $\{(\frac{3}{2}y - \partial_x - 1) \oplus 1, \partial_y(\frac{3}{2}y - \partial_x - 1) \oplus \partial_y\}$. Next applying η^{-1} , $H^0(\Delta \otimes_D Z^\bullet)$ is spanned by the images of $\{L_1 = (\partial_x + 2\partial_y - 1) \oplus 1, L_2 = -\frac{1}{2}(x\partial_x + 2y\partial_y + y\partial_x + 2x\partial_y - x - y) \oplus -\frac{1}{2}(x + y)\}$. Modulo the right ideal generated by $\{x - y, \partial_x + \partial_y\}$, we can re-express these cohomology classes by $\{(\partial_y - 1) \oplus 1, (y\partial_y - y - 1) \oplus -y\}$. Applying p_1 we get $\{L_{1,0} = \partial_y - 1, L_{2,0} = y\partial_y - y - 1\}$, which corresponds to a basis of $\text{Hom}_D(M, N)$ given by,

$$\begin{aligned}\phi_1 &: \frac{D}{D \cdot (\partial - 1)} \xrightarrow{\cdot[\partial - 1]} \frac{D}{D \cdot (\partial - 1)^2} \\ \phi_2 &: \frac{D}{D \cdot (\partial - 1)} \xrightarrow{\cdot[(x\partial - x - 1)]} \frac{D}{D \cdot (\partial - 1)^2}.\end{aligned}$$

6. EXTENSIONS OF HOLONOMIC D -MODULES

In this section we explain how one can modify our algorithm for the computation of $\text{Hom}_D(M, N)$ in order to compute explicitly the higher derived functors $\text{Ext}_D^i(M, N)$ for holonomic D -modules M and N .

A useful way to represent $\text{Ext}_D^i(M, N)$ is as the i -th Yoneda Ext group, which consists of equivalence classes of exact sequences,

$$\xi : 0 \rightarrow N \longrightarrow Q \longrightarrow X^{-i+2} \longrightarrow \dots \longrightarrow X^0 \longrightarrow M \longrightarrow 0,$$

for any list of (not necessarily free) D -modules Q, X^{-i+2}, \dots, X^0 . Two exact sequences ξ and ξ' are considered equivalent when there is a chain map of the form,

$$\begin{array}{ccccccccccc} \xi : 0 & \longrightarrow & N & \longrightarrow & Q & \longrightarrow & X^{-i+2} & \longrightarrow & \dots & \longrightarrow & X^0 & \longrightarrow & M & \longrightarrow & 0 \\ & & \downarrow \text{id}_N & & \downarrow & & \downarrow & & \dots & & \downarrow & & \downarrow \text{id}_M & & \\ \xi' : 0 & \longrightarrow & N & \longrightarrow & Q' & \longrightarrow & X'^{-i+2} & \longrightarrow & \dots & \longrightarrow & X'^0 & \longrightarrow & M & \longrightarrow & 0. \end{array}$$

In our modified algorithm we follow the same steps as in Algorithm 5.1, except that in Step 4 we compute $H^{-n+i}(\Lambda \otimes_{D_{2n}}^L (D_{2n}^{u_n}/P))$ instead of $H^{-n}(\Lambda \otimes_{D_{2n}}^L (D_{2n}^{u_n}/P))$. The output is a basis $\{\varphi_1, \dots, \varphi_k\}$ of the finite-dimensional K -vector space $H^i(\text{Hom}_D(X^\bullet, N))$, where X^\bullet is a free resolution of M ,

$$X^\bullet : 0 \rightarrow \underbrace{D^{r-a}}_{\text{degree } -a} \xrightarrow{\cdot M_{-a+1}} \dots \rightarrow D^{r-1} \xrightarrow{\cdot M_0} \underbrace{D^{r_0}}_{\text{degree } 0} \rightarrow M \rightarrow 0.$$

To obtain the i -th Yoneda Ext group from our output for $\text{Ext}_D^i(M, N)$, we follow the presentation of [18, Section 3.4] and associate to a cohomology class $\varphi \in H^i(\text{Hom}_D(X^\bullet, N))$ the exact sequence,

$$\xi(\varphi) : 0 \longrightarrow N \longrightarrow Q \longrightarrow D^{r-i+2} \longrightarrow \dots \longrightarrow D^{r_0} \rightarrow M \rightarrow 0.$$

Here, Q is the cokernel of $(\cdot M_{-i+1}, \varphi) : D^{r-i} \longrightarrow D^{r-i+1} \oplus N$, and the maps are all the natural ones. It is worth pointing out that N is indeed a submodule of Q by the following argument. $\phi \in H^i(\text{Hom}_D(X^\bullet, N))$ is computed as a map from X^{-i} to Y^0 with the property that $X^{-i-1} \rightarrow X^{-i} \rightarrow Y^0 \rightarrow N$ is zero. Assume $a \in \ker(N \rightarrow Q)$. Then $(0, a) \in \text{im}(\cdot M_{-i+1}, \varphi)$, so there is $b \in X^{-i}$ with $b \cdot M_{-i+1} = 0$ and $\varphi(b) = a$. Since $\ker(\cdot M_{-i+1}) = \text{im}(\cdot M_{-i})$, $b = c \cdot M_{-i}$ and it follows that $a = \varphi(c \cdot M_{-i}) = 0$.

Notice that the only difference between any $\xi(\varphi)$ and $\xi(\varphi')$ are their corresponding Q 's and the maps to and from them. In terms of the basis $\{\varphi_1, \dots, \varphi_k\}$ of $H^i(\text{Hom}_D(X^\bullet, N))$, the set of possible Q 's which appear can be packaged as the set,

$$V_i = \left\{ \frac{D^{r-i} \oplus D^{s_0}}{(0 \oplus N_0) + D \cdot \{(\cdot M_{-i} + \sum_{h=1}^k \kappa_h \varphi_h)(\vec{e}_j)\}_{j=1}^{r-i-1}} \mid (\kappa_1, \dots, \kappa_k) \in K^k \right\}.$$

When $i = 1$ for example, the 1-st Yoneda Ext group consists of equivalence classes of extensions of M by N ,

$$\xi : 0 \rightarrow N \rightarrow Q \rightarrow M = (D^{r_0}/D^{r-1} \cdot M_0) \rightarrow 0$$

where $Q = X^{-1} \oplus N$ modulo $X^{-2} \cdot M_{-1}$. Thus, once we have computed a basis $\{\varphi_1, \dots, \varphi_k\}$ of $H^1(\text{Hom}_D(X^\bullet, N))$ via the modified Algorithm 5.1, the possible extensions Q of M by N are,

$$V_1 = \left\{ \frac{D^{r-1} \oplus D^{s_0}}{(0 \oplus N_0) + D \cdot \{(\cdot M_{-1} + \sum_{h=1}^k \kappa_h \varphi_h)(\vec{e}_j)\}_{j=1}^{r-2}} \mid (\kappa_1, \dots, \kappa_k) \in K^k \right\}.$$

Example 6.1. Let $D = K\langle x, \partial \rangle$ be the first Weyl algebra and $M = D/D \cdot \partial$, $N = D/D \cdot x$. Then for Step 1 of Algorithm 5.1, we have the resolutions,

$$X^\bullet : 0 \rightarrow D^1 \xrightarrow{\cdot \partial} D^1 \rightarrow 0, \quad Y^\bullet : 0 \rightarrow D^1 \xrightarrow{\cdot x} D^1 \rightarrow 0$$

For Step 2, we form the complex $Z^\bullet = \text{Tot}(\tau(\text{Hom}_D(X^\bullet, D)) \boxtimes Y^\bullet)$,

$$Z^\bullet : 0 \leftarrow \underbrace{D_2^1}_{\text{degree 1}} \xleftarrow{\cdot \begin{bmatrix} \partial_x \\ y \end{bmatrix}} D_2^2 \xleftarrow{\cdot [y, -\partial_x]} D_2^1 \leftarrow 0$$

For Steps 3-6, we find that $H^1(\Delta \otimes_{D_2} Z^\bullet)$ is spanned by $\{1\}$, and projecting by p_1 , $\text{Ext}_D^1(M, N) \simeq H^1(\text{Hom}_D(X^\bullet, D/D \cdot x))$ is spanned by the natural projection $\varphi : D \rightarrow (D/D \cdot x)$. For $\kappa \in K$, the cohomology classes $\kappa\varphi$ correspond to the extensions on the bottom row of the following diagram,

$$\begin{array}{ccccccc} 0 \rightarrow & D & \xrightarrow{\cdot \partial} & D & \xrightarrow{\quad} & \frac{D}{D \cdot \partial} & \rightarrow 0 \\ & \downarrow \cdot \kappa & & \downarrow \cdot [0, 1] & & \downarrow \text{id}_{D/D \cdot \partial} & \\ 0 \rightarrow & \frac{D}{D \cdot x} & \xrightarrow{\cdot [1, 0]} & \frac{D \cdot \vec{e}_1 \oplus D \cdot \vec{e}_2}{D \cdot x\vec{e}_1 + D \cdot (\kappa\vec{e}_1 + \partial\vec{e}_2)} & \xrightarrow{\cdot \begin{bmatrix} 0 \\ 1 \end{bmatrix}} & \frac{D}{D \cdot \partial} & \rightarrow 0 \end{array}$$

When $\kappa \neq 0$, the module $Q(\kappa) = (D \cdot \vec{e}_1 \oplus D \cdot \vec{e}_2)/(D \cdot x\vec{e}_1 + D \cdot (\kappa\vec{e}_1 + \partial\vec{e}_2))$ is generated by \vec{e}_2 and is always isomorphic to $D/D \cdot x\partial$. When $\kappa = 0$, the module is no longer generated by \vec{e}_2 and is not isomorphic to $D/D \cdot x\partial$.

In fact, the module $(D \cdot \vec{e}_1 \oplus D \cdot \vec{e}_2)/(D \cdot x\vec{e}_1 + D \cdot (\kappa\vec{e}_1 + \partial\vec{e}_2))$ is always generated by the residue class of $\vec{e}_1 + \vec{e}_2$ and has the cyclic presentation $D/D \cdot \{\partial^2 x + \kappa x\partial, x^2\partial\}$ with respect to this generator. Using this presentation, the extensions take the form,

$$0 \rightarrow \frac{D}{D \cdot x} \xrightarrow{\cdot [-x\partial]} \frac{D}{D \cdot \{\partial^2 x + \kappa x\partial, x^2\partial\}} \xrightarrow{\cdot [x\partial+1]} \frac{D}{D \cdot \partial} \rightarrow 0.$$

One can picture $Q(\kappa)$ as the $K[x]$ -module $K[x] + x^{-1}K[x^{-1}]$ with the twisted multiplication rule $x \cdot (x^{-1}) = \kappa$ which is a direct sum if $\kappa = 0$.

7. ISOMORPHISM CLASSES OF D -MODULES

In this section, we give an algorithm to determine if two holonomic D -modules M and N are isomorphic and if so to produce an explicit isomorphism. For $M = N$, we also give an algorithm to find all isomorphisms from M to M and mention some well-known applications of the endomorphism ring $\text{End}_D(M)$. Here, $\text{End}_D(M)$ denotes the space of endomorphisms of a D -module $M = D^{r_M}/I_M$, where endomorphism means D -linear maps from M to M . Similarly, $\text{Iso}_D(M)$ denotes the units of the ring $\text{End}_D(M)$.

If holonomic M and N are isomorphic, then $\text{Hom}_D(M, N) \simeq \text{End}_D(M)$ is a finite-dimensional K -algebra. In the theory of finite dimensional K -algebras, the Jacobson radical J is the intersection of all maximal left ideals of E , and it has the property that the quotient E/J is a semi-simple K -algebra. By the Wedderburn-Artin theorem, a semi-simple algebra is isomorphic to a direct product of matrix rings over division algebras, and hence by taking the algebraic closure, we find that $E/J \otimes_K \overline{K}$ is isomorphic to a direct product of matrix rings over the field \overline{K} . One consequence of this decomposition is that the non-units of $E/J \otimes_K \overline{K}$ form a determinantal hypersurface. In particular, the units of $E/J \otimes_K \overline{K}$ form a Zariski open set, and hence the units of E/J also form a Zariski open set. Moreover, units and non-units respect the Jacobson radical in the sense that if j is in the Jacobson radical of E and if u is a unit of E then $u + j$ is also a unit, and similarly, if n is not a unit of E then $n + j$ is not a unit. We can thus conclude the following lemma.

Lemma 7.1. *Let M be a holonomic D -module. Then the space of D -linear isomorphisms $\text{Iso}_D(M)$ from M to itself is open in $\text{End}_D(M)$ under the Zariski topology.*

The lemma says that if holonomic M and N are isomorphic then most maps from M to N are isomorphisms. We now give an algorithm to determine if M and N are isomorphic based on Algorithm 5.1 and Lemma 7.1.

Algorithm 7.2. (Is M isomorphic to N ?)

INPUT: presentations $M \simeq D^{m_M}/D \cdot \{P_1, \dots, P_a\}$ and $N \simeq D^{m_N}/D \cdot \{Q_1, \dots, Q_b\}$ of left holonomic D -modules.

OUTPUT: “No” if $M \not\simeq N$; and “Yes” together with an isomorphism $\iota : M \rightarrow N$ if $M \simeq N$.

1. Compute bases $\{s_1, \dots, s_\sigma\}$ and $\{t_1, \dots, t_\tau\}$ for the vector spaces $V = \text{Hom}_D(M, N)$ and $W = \text{Hom}_D(N, M)$ using Algorithm 5.1, where s_i and t_j are respectively $m_M \times m_M$ and $m_N \times m_N$ matrices with entries in D representing homomorphisms by right multiplication. Recall that we view D^{m_M} and D^{m_N} as consisting of row vectors. If $\sigma \neq \tau$, return “No” and exit.
2. Introduce new indeterminates $\{\mu_i\}_1^\tau$ and $\{\nu_j\}_1^\tau$, and form the “generic homomorphisms” $\sum_i \mu_i s_i \in \text{Hom}_D(M, N)$ and $\sum_j \nu_j t_j \in \text{Hom}_D(N, M)$. Then the compositions $\sum_{i,j} \mu_i \nu_j s_i \cdot t_j : M \rightarrow N$ and $\sum_{i,j} \mu_i \nu_j t_j \cdot s_i : N \rightarrow M$ are respectively $m_M \times m_M$ and $m_N \times m_N$ -matrices with entries in $D[\mu_1, \dots, \mu_{m_M}, \nu_1, \dots, \nu_{m_N}]$.
3. Reduce the rows of the matrix $\sum_{i,j} \mu_i \nu_j s_i \cdot t_j - \text{id}_{m_M}$ modulo a Gröbner basis for $D \cdot \{P_1, \dots, P_a\} \subset D^{m_M}$. Force this reduction to be zero by setting the coefficients (which are inhomogeneous bilinear polynomials in μ_i, ν_j) of every

- standard monomial in every entry to be zero. Collect these relations in the ideal $I_M \subset K[\mu_1, \dots, \mu_{m_M}, \nu_1, \dots, \nu_{m_N}]$
4. Similarly, reduce the rows of the matrix $\sum_{i,j} \mu_i \nu_j t_j \cdot s_i - \text{id}_{m_N}$ modulo a Gröbner basis for $D \cdot \{Q_1, \dots, Q_b\} \subset D^{m_N}$. Force this reduction to be zero by setting the coefficients of every standard monomial in every entry to be zero, and collect these relations in the ideal $I_N \subset K[\mu, \nu]$.
 5. Put $I(V, W) = I_M + I_N \subset K[\mu, \nu]$. If $I(V, W)$ contains a unit, return “No” and exit.
 6. Otherwise compute an isomorphism $\sum_{i=1}^{\tau} k_i s_i$ in $\text{Hom}_D(M, N)$ by finding the first τ coordinates of any point in the zero locus of $I(V, W)$. For instance, we can do this by inductively finding $k_i \in K$ for each i from 1 to τ such that $I(V, W) + (\mu_1 - k_1, \dots, \mu_i - k_i)$ is a proper ideal. At each step i , this can be accomplished by trying different numbers for k_i until a suitable choice is found.
 7. Return “Yes” and the isomorphism $(\sum_{i=1}^{\tau} k_i s_i) : M \rightarrow N$.

Remark 7.3. Algorithm 7.2 can also be modified to detect whether M is a direct summand of N . Namely M is a direct summand of N if and only if the ideal I_M of step 3 is not the unit ideal. Similarly N is a direct summand of M if and only if the ideal I_N of step 4 is not the unit ideal.

Proof. Reduction of the generic matrix $\sum_{i,j} \mu_i \nu_j s_i \cdot t_j - \text{id}_{m_M}$ modulo $D \cdot \{P_1, \dots, P_a\}$ in step 3 leads to a generic remainder which depends on the parameters μ_i, ν_j . Moreover, since a Gröbner basis of $D \cdot \{P_1, \dots, P_a\}$ is parameter-free, this generic remainder has the property that its specialization to a fixed choice of parameters $\mu_i = a_i, \nu_j = b_j$ gives the remainder of $\sum_{i,j} a_i b_j s_i \cdot t_j - \text{id}_{m_M}$ modulo $D \cdot \{P_1, \dots, P_a\}$. Thus setting the remainder to zero in step 3 corresponds to deriving conditions on the parameters μ_i, ν_j which makes the endomorphism given by $\sum_{i,j} \mu_i \nu_j s_i \cdot t_j$ equal to the identity on M . This is possible if and only if M is a direct summand of N . The analogous statement holds for reduction of $\sum_{i,j} \mu_i \nu_j t_j \cdot s_i - \text{id}_{m_N}$ modulo $D \cdot \{Q_1, \dots, Q_b\}$ and setting its resulting remainder to zero. Here, setting a remainder to zero is equivalent to the vanishing of the coefficients of its standard monomials, and we collect these vanishing conditions in the ideal $I(V, W)$ of $K[\mu, \nu]$.

Now a linear combination $\sum_i a_i s_i : M \rightarrow N$ is an isomorphism with inverse $\sum_j b_j t_j : N \rightarrow M$ if and only if the composition $\sum_{i,j} a_i b_j s_i \cdot t_j$ is congruent to id_{m_M} modulo $D \cdot \{P_1, \dots, P_a\}$ and the opposite composition $\sum_{i,j} a_i b_j t_j \cdot s_i$ is congruent to id_{m_N} modulo $D \cdot \{Q_1, \dots, Q_b\}$. Thus the common zeroes $(a_1, \dots, a_{\tau}, b_1, \dots, b_{\tau})$ of $I(V, W)$ correspond to isomorphisms $\sum_i a_i s_i$ and their inverses $\sum_j b_j t_j$. In particular, if $I(V, W)$ is the entire ring, which we detect by searching for 1 in a Gröbner basis of $I(V, W)$, then there are no isomorphisms.

On the other hand if $I(V, W)$ is proper, then M and N are isomorphic and we obtain an explicit isomorphism from finding any common solution of $I(V, W)$. By Lemma 7.1, the invertible homomorphisms from M to N are Zariski dense in the vector space $\text{Hom}_D(M, N)$. Hence, a common solution can be explicitly found by intersecting the zero locus of $I(V, W)$ with a suitable number of generic hyperplanes $\{\mu = k_i\}$. Because of denseness, each of these hyperplanes can be found in a finite number of steps. In other words, if $I(V, W) + \langle \mu_1 - k_1, \dots, \mu_{i-1} - k_{i-1} \rangle$ is proper, then

there are only finitely many k_i for which the sum $I(V, W) + \langle \mu_1 - k_1, \dots, \mu - k_i \rangle$ is the unit ideal. \square

Remark 7.4. Once we have specialized the μ_i in a common solution of $I(V, W)$, then the ν_j are determined because of the bilinear nature of the relations (which gives linear relations for the ν_j once all μ_i are chosen). This also means that if there is any solution, then the μ_i are rational functions in the ν_j and vice versa. In particular, if $\phi \in \text{Hom}_D(M, N)$ is defined over the field K then ϕ^{-1} is defined over K as well and no field extensions are required. We now give two simple examples, one where M and N are isomorphic, and one where they are not.

Example 7.5. Let $n = 1$ and $M = N = D/D \cdot \partial^2$. One checks that $V = W = \text{Hom}_D(M, N)$ is generated by the 4 morphisms $s_1 = \cdot(\partial)$, $s_2 = \cdot(x\partial)$, $s_3 = \cdot(1)$, and $s_4 = \cdot(x^2\partial - x)$. We obtain the generic morphism

$$\begin{aligned} \sum_{i=1}^4 \sum_{j=1}^4 \mu_i \nu_j t_j \cdot s_i - 1 &= (\mu_3 \nu_3 - \mu_1 \nu_4 - 1) \\ &\quad + (-\mu_4 \nu_3 - \mu_2 \nu_4 - \mu_3 \nu_4)x \\ &\quad + (\mu_3 \nu_1 + \mu_1 \nu_2 + \mu_1 \nu_3)\partial \\ &\quad + (-\mu_4 \nu_1 + \mu_2 \nu_2 + \mu_3 \nu_2 + \mu_2 \nu_3 + \mu_1 \nu_4)x\partial \\ &\quad + (\mu_4 \nu_3 + \mu_2 \nu_4 + \mu_3 \nu_4)x^2\partial \end{aligned}$$

plus 9 other terms which are in $D \cdot \partial^2$ independently of the parameters.

Hence in order for $\sum_{i=1}^4 \mu_i s_i$ to be an isomorphism, the μ_i need to be part of a solution to the ideal

$$\begin{aligned} I(V, W) &= (\mu_3 \nu_3 - \mu_1 \nu_4 - 1, \\ &\quad -\mu_4 \nu_3 - \mu_2 \nu_4 - \mu_3 \nu_4, \\ &\quad \mu_3 \nu_1 + \mu_1 \nu_2 + \mu_1 \nu_3, \\ &\quad -\mu_4 \nu_1 + \mu_2 \nu_2 + \mu_3 \nu_2 + \mu_2 \nu_3 + \mu_1 \nu_4, \\ &\quad \mu_4 \nu_3 + \mu_2 \nu_4 + \mu_3 \nu_4). \end{aligned}$$

This ideal is not the unit ideal and has degree 8. Hence there are isomorphisms between M and N . Pick “at random” $\mu_1 = 1$, $\mu_2 = 2$, and $\mu_3 = 0$. Then the ideal $I(V, W) + (\mu_1 - 1, \mu_2 - 2, \mu_3 - 0)$ equals the ideal $(\mu_1 - 1, \mu_2 - 2, \mu_3, \nu_4 + 1, \nu_2 + \nu_3, \nu_1 + \frac{1}{2}\nu_3, \mu_4 \nu_3 - 2)$. We see that we have to avoid $\mu_4 = 0$ but otherwise have complete choice.

Example 7.6. Let $n = 1$, $M = D/D \cdot \partial^2$, and $N = D/D \cdot \partial$. One checks that $V = \text{Hom}_D(N, M)$ is generated by $t_1 = \cdot(\partial)$ and $t_2 = \cdot(x\partial - 1)$ while $W = \text{Hom}_D(M, N)$ is generated by $s_1 = \cdot(1)$ and $s_2 = \cdot(x)$. The sum $\sum \mu_i \nu_j s_i \cdot t_j$ takes the form

$$\mu_2 \nu_2 x^2 \partial + (\mu_1 \nu_2 + \mu_2 \nu_1) x \partial + \mu_1 \nu_1 \partial - (\mu_1 \nu_2 + \mu_2 \nu_2).$$

Modulo $D \cdot \partial$ we want this to be 1, so we get the relation

$$\mu_2 \nu_1 - \mu_1 \nu_2 = 1.$$

We note that this equation has plenty of solutions, which means that M can be realized as a summand of N . On the other hand, the sum $\sum \mu_i \nu_j t_j \cdot s_i$ takes the

form

$$\mu_1\nu_1\partial + (\mu_1\nu_2 + \mu_2\nu_1)x\partial - \mu_1\nu_2 - \mu_2\nu_2x + \mu_2\nu_2x^2\partial.$$

Modulo $D \cdot \partial^2$ we want this to be 1, so we get the relations

$$\begin{aligned} -\mu_1\nu_2 &= 1, \\ \mu_1\nu_1 &= 0, \\ \mu_1\nu_2 + \mu_2\nu_1 &= 0, \\ \mu_2\nu_2 &= 0. \end{aligned}$$

Putting all the equations together, we obtain the unit ideal, and hence M and N are not isomorphic.

For M and N isomorphic, we now give a method to find all possible isomorphisms, that is we study the units $\text{Iso}_D(M)$ in the endomorphism ring $V = \text{End}_D(M)$.

Lemma 7.7. *The isomorphism set $\text{Iso}_D(M)$ is a smooth affine variety which is connected if K is algebraically closed.*

Proof. As we have seen, $\text{Iso}_D(M)$ is isomorphic to the nonempty affine variety $\text{Var}(M) = \text{Var}(I(V, V))$ defined in the variables μ_i, ν_j . Here, a point of $\text{Var}(M)$ with coordinates $(\mu_1, \dots, \mu_\tau, \nu_1, \dots, \nu_\tau)$ corresponds to the isomorphism $\sum_{i=1}^\tau \mu_i s_i$. Now any isomorphism $\phi : M \rightarrow M$ induces an isomorphism from the variety to itself, sending (μ, ν) to (μ', ν') where $\sum \mu_i s_i \circ \phi = \sum \mu'_i s_i$. This action is regular in ϕ (since we showed μ is rational in ν), and transitive since $\psi \in \text{Hom}_D(M, M)$ equals $(\psi \circ \phi^{-1}) \circ \phi$. It follows that $\text{Var}(M)$ is a smooth variety because it is a homogeneous space over itself via a transitive action. As we have seen in Lemma 7.1, $\text{Iso}(M)$ is Zariski open in $\text{End}_D(M)$ (which is an affine space and therefore normal) and hence connected if K is algebraically closed. \square

Since the isomorphisms $\text{Iso}_D(M)$ are Zariski open in $\text{End}_D(M)$, one can ask for a method to compute the equations defining the complementary closed set of non-isomorphisms.

Definition 7.8. The ideal in $K[\nu]$ that determines the closed set $\text{End}_D(M) \setminus \text{Iso}_D(M)$ of non-isomorphisms of M is called $\Delta(M)$, the *defect ideal*.

Algorithm 7.9. (Computing the defect ideal)

INPUT: Generators for a holonomic D -module M .

OUTPUT: The defect ideal $\Delta(M)$ defining the non-isomorphisms of $\text{End}_D(M)$.

1. Perform Steps 1 through 4 of Algorithm 7.2 with $M = N$ as input to obtain the ideal $I(V, V) \subset K[\{\mu_i\} \cup \{\nu_j\}]$.
2. Regard each of the ζ generators of $I(V, V)$ as a linear inhomogeneous equation in the variables μ_i with coefficients involving ν_j as parameters, and collect all these equations in a single matrix equation $A \cdot \mu = b$, $A \in K[\nu]^{\zeta \times \tau}$.
3. Compute all $\tau \times \tau$ minors of A and collect them in an ideal $\Delta(M) \subset K[\nu]$.
4. Return $\Delta(M)$.

Proof. A point ν corresponds to an isomorphism with inverse μ if and only if the system $A \cdot \mu = b$ has exactly one solution for μ . This is equivalent to the $\zeta \times \tau$ matrix A having rank τ and the augmented matrix $(A|b)$ also having rank τ . The matrix A will have rank τ if and only if any one of its $\tau \times \tau$ minors is nonzero. Similarly, the augmented matrix $(A|b)$ will also have rank τ if in addition all $(\tau + 1) \times (\tau + 1)$

minors vanish. We claim that each $(\tau + 1) \times (\tau + 1)$ minor of $(A|b)$ must actually be identically zero. Otherwise it would impose an algebraic condition which must be satisfied by the isomorphisms in the coordinates ν of $\text{End}_D(M)$. But this cannot happen since the isomorphisms are an open set by Lemma 7.1. Thus we have shown that the space of non-isomorphisms ν is defined by the equations obtained from the vanishing of all $\tau \times \tau$ minors of A . \square

Remark 7.10. In Lemma 7.1, we saw that modulo the Jacobson radical, which is a linear subspace, then $\text{End}_D(M)$ is the product of simple K -algebras. Moreover when K is algebraically closed, then a simple K -algebra is a matrix algebra. It follows that in an algebraic closure of K , the radical of $\Delta(M)$ is generated by linear forms corresponding to the Jacobson radical and a single determinant which is the product of the determinants of the matrix algebras. However, we have not yet understood what happens when the field of input K is not algebraically closed. Optimistically, we hope that Algorithm 7.9 produces an ideal $\Delta(M)$ whose radical over K is also generated by linear forms and a single determinant. The following example shows at least that $\Delta(M)$ might not be radical.

Example 7.11. Let us look at our Example 7.5. There the system of equations $\{-\mu_1\nu_4 + \mu_3\nu_3 = 1, -\mu_2\nu_4 - \mu_3\nu_4 - \mu_4\nu_3 = 0, \mu_1(\nu_2 + \nu_3) + \mu_3\nu_1 = 0, \mu_1\nu_4 + \mu_2(\nu_2 + \nu_3) + \mu_3\nu_2 - \mu_4\nu_1 = 0\}$ can be rewritten in the form $A \cdot \mu = b$ as,

$$\begin{pmatrix} -\nu_4 & & \nu_3 & \\ & -\nu_4 & -\nu_4 & -\nu_3 \\ \nu_2 + \nu_3 & & \nu_1 & \\ \nu_4 & \nu_2 + \nu_3 & \nu_2 & -\nu_1 \end{pmatrix} \begin{pmatrix} \mu_1 \\ \mu_2 \\ \mu_3 \\ \mu_4 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

In order to assure that A has full rank we need the determinant

$$\Delta(M) = \nu_2^2\nu_3^2 + 2\nu_2\nu_3^3 + \nu_3^4 + 2\nu_1\nu_2\nu_3\nu_4 + 2\nu_1\nu_3^2\nu_4 + \nu_1^2\nu_4^2$$

to be nonzero. We conclude that the locus of not invertible morphisms is given by the vanishing of the determinant of $\Delta(M)$. Note also that $\Delta(M) = (\nu_2\nu_3 + \nu_3^2 + \nu_1\nu_4)^2$ which in particular is not square free.

We end by discussing the endomorphism ring $E = \text{End}_D(M)$.

Remark 7.12. A well-known application of $E = \text{End}_D(M)$ is towards decompositions of M . By the Krull-Schmidt-Azumaya theorem, a D -module M has a decomposition into a direct sum of indecomposable submodules (meaning that they cannot be further decomposed into a direct sum of nonzero submodules), and any such decomposition is unique up to re-ordering and isomorphism (see e.g. [8, Theorem 19.21]). There is a bijective correspondence between (1) the decompositions of M into a direct sum of submodules and (2) the decompositions of the identity element $1 = e_1 + \dots + e_s$ of E into pairwise orthogonal idempotents [3, Theorem 1.7.2]. The correspondence is gotten by taking a set of orthogonal idempotents $\{e_1, \dots, e_s\}$ and producing the decomposition $M = e_1 \cdot M \oplus \dots \oplus e_s \cdot M$. Thus, an algorithm which produces a full set of orthogonal idempotents for the K -algebra $\text{End}_D(M)$ combined with Algorithm 5.1 would give a method to decompose holonomic D -modules into indecomposables.

Computation in finite-dimensional K -algebras E has recently been an area of active research. When K is a number field, early work of Friedl and Ronyai provides polynomial-time algorithms to decompose E into simple algebras if E is semi-simple

and to find the radical of E in general [5]. When $K = \mathbb{C}$, Eberly has given Las Vegas polynomial time algorithms to find the decomposition of a simple algebra as a full matrix ring [4]. We should also mention that the radical of E is independent of field extension of K while the decomposition into simple algebras depends upon the field K . Thus if we are willing to use $K = \mathbb{C}$, then a full set of orthogonal idempotents can indeed be algorithmically computed.

Let us also describe another method based on computational algebraic geometry to obtain information on the invariants d_i in the decomposition

$$E/\text{Jac}(E) \otimes_K \bar{K} = \prod_1^d \text{End}_{\bar{K}}(\bar{K}^{d_i})$$

where \bar{K} denotes the algebraic closure of K . We will compute the de Rham cohomology groups of the complement of $\text{Var}(\Delta(M))$ in $\text{End}_D(M) = \mathbb{C}^\tau = \text{Spec}(\mathbb{C}[\nu])$ using the algorithm developed by the second author in [16]. This algorithm will in some sense allow us to pretend that K is already algebraically closed. Namely, the algorithm can be used on input defined over any computable subfield of the complex numbers, but always computes $\dim_{\mathbb{C}}(H_{dR}^\bullet(\mathbb{C}^n \setminus Y, \mathbb{C}))$. What we now need is a method that enables us to sort out the d_i from the Betti numbers of $\mathbb{C}^\tau \setminus \text{Var}(\Delta(M))$.

Consider $E \otimes_K \mathbb{C}$. As we have shown, its units are (homotopy equivalent to) the units of a product of the form $\prod_{i=1}^d \text{End}_{\mathbb{C}}(\mathbb{C}^{d_i})$. The non-isomorphisms in each factor are given by the vanishing of the appropriate determinant, and the isomorphisms are just the elements of the general linear group $Gl(d_i, \mathbb{C})$.

The cohomology of $Gl(n, \mathbb{C})$ is well understood and best expressed for our purposes in terms of the Poincaré polynomial.

Definition 7.13. Let h_0, \dots, h_l, \dots be the dimensions of the de Rham cohomology groups of a complex manifold T . Then the *Poincaré series (polynomial)* $P_T(q)$ is defined by

$$P_T(q) = \sum_{i \geq 0} h_i q^i.$$

The Poincaré polynomial behaves very nicely under products $M_2 = M_1 \times M_3$ of manifolds:

$$P_{M_1}(q) \cdot P_{M_3}(q) = P_{M_2}(q).$$

An old result ([19], Theorems 7.11.A and 8.16.B) states that $P_{Gl(d_i, \mathbb{C})}(q) = \prod_{j=1}^{d_i} (1 + t^{2j-1})$. Hence the Poincaré polynomial of a product of general linear groups $\prod_{i=1}^d P_{Gl(d_i, \mathbb{C})}(q)$ equals

$$(11) \quad \prod_{i=1}^d \prod_{j=1}^{d_i} (1 + t^{2j-1}) = (1 + t)^{\sum_{d_i > 0} 1} \cdot (1 + t^3)^{\sum_{d_i > 1} 1} \cdot (1 + t^5)^{\sum_{d_i > 2} 1} \cdot \dots$$

Thus in order to compute the d_i one then has the following algorithm.

Algorithm 7.14.

INPUT: Generators and relations for the left module M .

OUTPUT: The invariants d_i associated to $\text{End}_D(M)$.

1. Compute the defect ideal $\Delta(M) \subseteq K[\nu]$ by using Algorithm 7.9.
2. Compute the dimensions $h_k = \dim_{\mathbb{C}}(H_{dR}^k(\mathbb{C}^n \setminus \text{Var}(\Delta(M))))$.

3. Factor the polynomial $P_{\text{Iso}_D(M)}(q) := \sum h_k q^k$ into

$$(1+q)^{k_1} \cdot (1+q^3)^{k_2} \cdot \dots \cdot (1+q^{2l-1})^{k_l}.$$

4. Compute the d_i by comparing the expansion from the previous item with equation (11). Return the d_i .

Example 7.15. Continuing our Example 7.11 we compute the de Rham cohomology groups of the complement of $\text{Var}(\nu_2^2\nu_3^2+2\nu_2\nu_3^3+\nu_3^4+2\nu_1\nu_2\nu_3\nu_4+2\nu_1\nu_3^2\nu_4+\nu_1^2\nu_4^2)$ in \mathbb{C}^4 . Using Macaulay2 one obtains $h_0 = h_1 = h_3 = h_4 = 1$ and all other h_k vanish. Then the Poincaré polynomial is $1+q+q^3+q^4 = (1+q)(1+q^3)$. This means that $d = 1$ and $d_1 = 2$.

8. APPENDIX

In this section, we provide a short survey of the ideas that lead to an algorithm for restriction and then, mostly for purposes of reference here and otherwise, list an algorithm to compute integration. All the main ideas are taken from [10, 11, 16]

Definitions 8.1. Fix an integer d with $0 \leq d \leq n$ and set $H = \text{Var}(x_1, \dots, x_d)$. For $\alpha \in \mathbb{Z}^n$, we set $\alpha_H = (\alpha_1, \dots, \alpha_d, 0, \dots, 0)$.

On the ring D we define the V_d -filtration $F_H^k(D)$ as the K -linear span of all operators $x^\alpha \partial^\beta$ for which $|\alpha_H| + k \geq |\beta_H|$. More generally, on a free D -module $A = \oplus_{j=1}^t D \cdot e_j$ we define

$$F_H^k(A)[\mathbf{m}] = \sum_{j=1}^t F_H^{k-\mathbf{m}(j)}(D) \cdot e_j,$$

where \mathbf{m} is an element of \mathbb{Z}^m . We shall call \mathbf{m} the *shift vector*. A shift vector is tied to a fixed set of generators.

We define the V_d -degree of an operator $P \in A[\mathbf{m}]$, $V_d \deg(P[\mathbf{m}])$, to be the smallest k such that $P \in F_H^k(A[\mathbf{m}])$.

If M is a quotient of the free D -module $A = \oplus_1^t D \cdot e_j$, $M = A/I$, we define the V_d -filtration on M by $F_H^k(M[\mathbf{m}]) = F_H^k(A[\mathbf{m}]) + I$. For submodules N of A we define the V_d -filtration by intersection: $F_H^k(N[\mathbf{m}]) = F_H^k(A[\mathbf{m}]) \cap N$.

If M is a submodule of the free module $A[\mathbf{m}]$, then a V_d -strict Gröbner basis or a V_d -Gröbner basis for M is a set of generators $\{m_1, \dots, m_\tau\}$ for M which satisfies: for all $m \in M$ we can find $\{\alpha_i\}_1^\tau \in D$ such that $m = \sum \alpha_i m_i$ and $V_d \deg(\alpha_i m_i[\mathbf{m}]) \leq V_d \deg(m[\mathbf{m}])$ for all i .

Definitions 8.2. A complex of free D -modules $\dots \rightarrow A^{i-1} \xrightarrow{\phi^{i-1}} A^i \xrightarrow{\phi^i} A^{i+1} \rightarrow \dots$ is said to be V_d -adapted at A^i with respect to certain shift vectors $\mathbf{m}_{i-1}, \mathbf{m}_i, \mathbf{m}_{i+1}$ if

$$\phi^i(F_H^k(A^i[\mathbf{m}_i])) \subseteq F_H^k(A^{i+1}[\mathbf{m}_{i+1}])$$

and also

$$\phi^{i-1}(F_H^k(A^{i-1}[\mathbf{m}_{i-1}])) \subseteq F_H^k(A^i[\mathbf{m}_i])$$

for all k .

We shall say that the complex is V_d -strict at A^i if it is V_d -adapted at A^i and moreover

$$\text{im}(\phi^{i-1}) \cap F_H^k(A^i[\mathbf{m}_i]) = \text{im}(\phi^{i-1}|_{F_H^k(A^{i-1}[\mathbf{m}_{i-1}]})$$

for all k .

For $1 \leq d \leq n$ we set $\theta_d = x_1 \partial_1 + \dots + x_d \partial_d$ and $\theta_0 = 0$. Recall that a D -module $M[\mathbf{m}] = A[\mathbf{m}]/I$ is called *specializable to H* if there is a polynomial $b(s)$ in a single variable such that

$$(12) \quad b(\theta_d + k) \cdot F_H^k(M[\mathbf{m}]) \subseteq F_H^{k-1}(M[\mathbf{m}])$$

for all k (cf. [11]). Introducing

$$\mathrm{gr}_H^k(M[\mathbf{m}]) = (F_H^k(M[\mathbf{m}])) / (F_H^{k-1}(M[\mathbf{m}])),$$

this can be written as

$$b(\theta_d + k) \cdot \mathrm{gr}_H^k(M[\mathbf{m}]) = 0.$$

The monic polynomial $b(\theta)$ of least degree satisfying an equation of the type (12) is called the *b -function for restriction of $M[\mathbf{m}]$ to H* .

Remark 8.3. Specializability descends to quotients and submodules. Namely, assume that $M[\mathbf{m}] = (A/I)[\mathbf{m}]$ is specializable and $N[\mathbf{m}] = (A'/I)[\mathbf{m}]$ is a submodule of M (where $I \subseteq A' \subseteq A$). Let $b(s)$ be a polynomial that satisfies $b(\theta_d + k) \cdot F_H^k(A[\mathbf{m}]) \subseteq F_H^{k-1}(A[\mathbf{m}]) + I$. Then clearly $b(\theta_d + k) \cdot F_H^k(A[\mathbf{m}]) \subseteq F_H^{k-1}(A[\mathbf{m}]) + A'$ as well and hence $(M/N)[\mathbf{m}]$ is specializable to H . On the other hand, if $P' \in F_H^k(A'[\mathbf{m}]) = F_H^k(A[\mathbf{m}]) \cap A'$ then $b(\theta + k) \cdot P = Q + Q'$ where $Q \in F_H^{k-1}(A[\mathbf{m}])$ and $Q' \in I$ and hence $Q \in F_H^{k-1}(A[\mathbf{m}]) \cap A' = F_H^k(A'[\mathbf{m}])$. This implies that N is also specializable and we see that the b -functions for restriction of $N[\mathbf{m}]$ and for $(M/N)[\mathbf{m}]$ divide the b -function for restriction of $M[\mathbf{m}]$ to H .

Notice that independently of d , $\mathrm{gr}_H^\bullet(D[0]) \cong D$, as a ring.

It has been shown by Oaku and Takayama in [10] (Proposition 3.8 and following remarks) how to compute V_d -strict Gröbner bases, and for any D -module M positioned in degree b a free V_d -strict resolution $(A^\bullet[\mathbf{m}_\bullet], \phi^\bullet)$ of $M[\mathbf{m}_b]$, $A^i = \bigoplus_1^{r_i} D$, $r_i = 0$ if $i > b$. The construction given in [10] allows for arbitrary \mathbf{m}_b .

The method employed is to construct a free resolution with the usual technique of finding a Gröbner basis for $\ker(A^i \rightarrow A^{i+1})$ and calculating the syzygies on this basis. The trick is to impose an order that refines the partial ordering given by V_d -degree, together with a homogenization technique.

In [16] was given an algorithm to compute V_n -strict resolutions for right bounded complexes, based on Eilenberg-MacLane resolutions. In [17] an improved algorithm is given that usually computes a much smaller resolution and is also easier to implement. We shall assume that the reader is familiar with the techniques from [10, 11, 16, 17].

An idea first stated in [11] and further developed in [16] yields a theorem which in order to state we need to introduce some more terminology for.

Definition 8.4. Let $\tilde{\Omega}_d = D/\{x_1, \dots, x_d\} \cdot D$ and $\Omega_d = D/\{\partial_1, \dots, \partial_d\} \cdot D$.

The *restriction of the complex $A^\bullet[\mathbf{m}_\bullet]$ to the subspace H* is the complex $\tilde{\Omega}_d \otimes_D^L A^\bullet[\mathbf{m}_\bullet]$ considered as a complex in the category of $K\langle x_{d+1}, \partial_{d+1}, \dots, x_n, \partial_n \rangle$ -modules.

The *integration of $A^\bullet[\mathbf{m}_\bullet]$ along H* is the complex $\Omega_d \otimes_D^L A^\bullet[\mathbf{m}_\bullet]$ considered as a complex in that same category.

We need to make a convention about the V_d -filtration on tensor products over D .

Definition 8.5. If $A[\mathbf{m}]$ is a free H -graded D -module with shift vector \mathbf{m} then $\tilde{\Omega}_d \otimes_D A[\mathbf{m}]$ is filtered by $F_H^k(\tilde{\Omega}_d \otimes_D A[\mathbf{m}]) :=$ the K -span of $\{\bar{P} \otimes_D Q | V_d \deg(P) + V_d \deg(Q[\mathbf{m}]) \leq k\}$. Note that as $\tilde{\Omega}_d$ equals $K[\partial_1, \dots, \partial_d] \langle x_{d+1}, \partial_{d+1}, \dots, x_n, \partial_n \rangle$ as right D -module, $F_H^k(\tilde{\Omega}_d \otimes_D A[\mathbf{m}])$ equals the free $K \langle x_{d+1}, \partial_{d+1}, \dots, x_n, \partial_n \rangle$ -module on the symbols $\{(P_1, \dots, P_{\text{rk}_D(A)}) | P_j \in K[\partial_1, \dots, \partial_d], \deg_{\partial}(P_j) \leq k - \mathbf{m}(j) \forall j\}$.

If $A^\bullet[\mathbf{m}_\bullet]$ is a V_d -strict complex, we denote by $F_H^k(\tilde{\Omega}_d \otimes A^\bullet[\mathbf{m}_\bullet])$ the complex whose modules are the $F_H^k(\tilde{\Omega}_d \otimes A^i[\mathbf{m}_i])$ as defined above, and the maps are induced from A^\bullet .

Algorithm 8.6. Let $(A^\bullet[\mathbf{m}_\bullet], \delta^\bullet)$ be a V_d -strict complex of free D -modules. The restriction of $A^\bullet[\mathbf{m}_\bullet]$ to $H = \text{Var}(x_1, \dots, x_d)$, interpreted as a complex of modules over $K \langle x_{d+1}, \partial_{d+1}, \dots, x_n, \partial_n \rangle$, can be computed as follows:

1. Compute the b -function $b_{A^\bullet[\mathbf{m}_\bullet]}(s)$ for restriction of $A^\bullet[\mathbf{m}_\bullet]$ to H .
2. Find integers k_0, k_1 with $(b_{A^\bullet[\mathbf{m}_\bullet]}(k) = 0, k \in \mathbb{Z}) \Rightarrow (k_0 \leq k \leq k_1)$.
3. $\tilde{\Omega}_d \otimes_D^L A^\bullet$ is quasi-isomorphic to the complex

$$(13) \quad \dots \rightarrow \frac{F_H^{k_1}(\tilde{\Omega}_d \otimes_D A^i[\mathbf{m}_i])}{F_H^{k_0-1}(\tilde{\Omega}_d \otimes_D A^i[\mathbf{m}_i])} \rightarrow \frac{F_H^{k_1}(\tilde{\Omega}_d \otimes_D A^{i+1}[\mathbf{m}_{i+1}])}{F_H^{k_0-1}(\tilde{\Omega}_d \otimes_D A^{i+1}[\mathbf{m}_{i+1}])} \rightarrow \dots$$

This is a complex of free finitely generated $K \langle x_{d+1}, \partial_{d+1}, \dots, x_n, \partial_n \rangle$ -modules. \square

Let us now consider the question how to compute $\Omega_d \otimes_D^L C^\bullet$. This problem is intimately related to the restriction algorithm. The reason is the *Fourier transform*, which is an algebra automorphism from D to itself and defined as follows:

$$\mathcal{F}_d(x_i) = \partial_i, \quad \mathcal{F}_d(\partial_i) = -x_i.$$

The perhaps surprising minus sign is required to keep the Leibniz relation $x_i \partial_i + 1 = \partial_i x_i$ intact. The Fourier transform can be used to define an equivalence of the category of left D -modules with itself via “extension of scalars”: $\mathcal{F}_d(M) := D \otimes_D M$ where D is on the left considered as just D while on the right D acts on D via \mathcal{F}_d . So for example if $1 \otimes m \in \mathcal{F}_d(M)$ then $x_i \cdot 1 \otimes m = x_i \otimes m = 1 \otimes (-\partial_i \cdot m)$.

If we apply \mathcal{F}_d to the integration problem we are reduced to computing the restriction of $\mathcal{F}_d(C^\bullet)$ since $\mathcal{F}_d(\Omega_d) = \tilde{\Omega}_d$. We are led to introduce therefore a \tilde{V}_d -filtration which is defined by

$$\tilde{F}_H^k(D) = \mathcal{F}_d(F_H^k(D)).$$

This extends just as the V_d -filtration to submodules and quotients of shifted free modules. One also defines a b -function for integration of the complex C^\bullet , \tilde{b}_{C^\bullet} , as the b -function for restriction of the complex $\mathcal{F}_d(C^\bullet)$. Let us illustrate this concept with an

Example 8.7. Suppose $n = 2$, $M = D_2/D_2 \cdot (\partial_1, \partial_2) \cong K[x_1, x_2]$ and $d = 1$. The b -function $b(\theta_1)$ for restriction of the complex X^\bullet with $X^0 = M$ and $X^i = 0$ otherwise is $b(\theta_1) = \theta_1$ because $x_1 \partial_1 \cdot F_H^0(D_2) \subseteq F_H^{-1}(D_2) + D_2 \cdot \{\partial_1, \partial_2\}$. On the other hand, the b -function $\tilde{b}(\theta_1)$ for integration is $\theta_1 + 1$ because $(x_1 \partial_1 + 1) \cdot \tilde{F}_H^0(D_2) \subseteq \tilde{F}_H^{-1}(D_2) + D_2 \cdot \{x_1, x_2\}$.

Theorem 8.6 implies then the following algorithm.

Algorithm 8.8. Let $(A^\bullet[\mathbf{m}_\bullet], \delta^\bullet)$ be a \tilde{V}_d -strict complex of free D -modules. The integration of $A^\bullet[\mathbf{m}_\bullet]$ along $\partial_1, \dots, \partial_d$, interpreted as a complex of modules over $K \langle x_{d+1}, \partial_{d+1}, \dots, x_n, \partial_n \rangle$, can be computed as follows:

1. Compute the b -function $\tilde{b}_{A^\bullet[\mathbf{m}_\bullet]}(s)$ for integration of $A^\bullet[\mathbf{m}_\bullet]$ along H .
2. Find integers k_0, k_1 with $(b_{A^\bullet[\mathbf{m}_\bullet]}(k) = 0, k \in \mathbb{Z}) \Rightarrow (k_0 \leq k \leq k_1)$.
3. $\Omega_d \otimes_D^L A^\bullet$ is quasi-isomorphic to the complex

$$(14) \quad \cdots \rightarrow \frac{\tilde{F}_H^{k_1}(\Omega_d \otimes_D A^i[\mathbf{m}_i])}{\tilde{F}_H^{k_0-1}(\Omega_d \otimes_D A^i[\mathbf{m}_i])} \rightarrow \frac{\tilde{F}_H^{k_1}(\Omega_d \otimes_D A^{i+1}[\mathbf{m}_{i+1}])}{\tilde{F}_H^{k_0-1}(\Omega_d \otimes_D A^{i+1}[\mathbf{m}_{i+1}])} \rightarrow \cdots$$

This is a complex of free finitely generated $K\langle x_{d+1}, \partial_{d+1}, \dots, x_n, \partial_n \rangle$ -modules.

Example 8.9. $K[x_1, x_2]$ has a \tilde{V}_1 -strict free resolution

$$D[-1] \xrightarrow{\cdot[\partial_1, \partial_2]} D \oplus D[0, -1] \xrightarrow{\cdot \begin{bmatrix} \partial_2 \\ -\partial_1 \end{bmatrix}} D[0].$$

Continuing our example, we want to find k_1, k_0 satisfying the condition $((k+1) = 0, k \in \mathbb{Z}) \Rightarrow k_0 \leq k \leq k_1$. Clearly $k_0 = k_1 = -1$ should be chosen.

Then the integration of M along ∂_1 is, according to the theorem, quasi-isomorphic to the complex

$$\cdots \rightarrow 0 \rightarrow \frac{\tilde{F}_H^{-1}(D_2[-1])}{\tilde{F}_H^{-2}(D_2)} \rightarrow \frac{\tilde{F}_H^{-1}(D_2 \oplus D_2[0, -1])}{\tilde{F}_H^{-2}(D_2)} \rightarrow \frac{\tilde{F}_H^{-1}(D_2[0])}{\tilde{F}_H^{-2}(D_2)} \rightarrow 0 \rightarrow \cdots$$

Since $\tilde{F}_H^{-1}(D_2)$ is the span of all monomials of D_2 with positive V_1 -degree and $\tilde{F}_H^{-2}(D_2)$ is spanned by those of V_1 -degree exceeding 1, the complex above is (with $D_1 = K\langle x_2, \partial_2 \rangle$)

$$\cdots \rightarrow D_1 \cdot 1 \xrightarrow{\cdot \partial_2} D_1 \cdot 1 \oplus 0 \rightarrow 0 \rightarrow 0 \rightarrow \cdots$$

the cohomology of which is exactly $K[x_2]$, shifted cohomologically by one relative to the input.

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